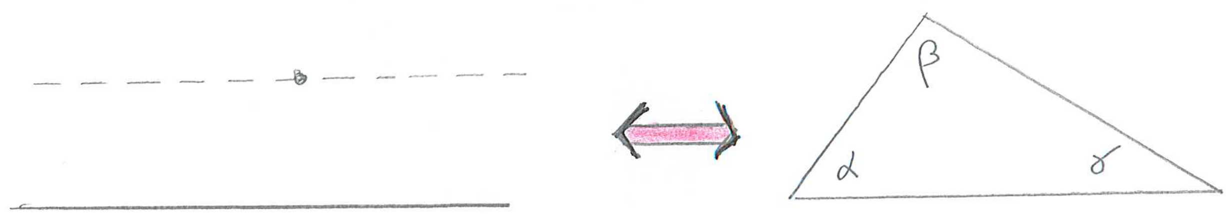
INTRODUCTION

In this course we will generally follow arguments presented in the text. However, we will also provide different derivations which are equivalent, but which may have a different starting point.

Example: Euclidean & Non-Euclidean Geometry



$$\alpha+\beta+\gamma=180^\circ$$

Euclid's postulate about parallel lines is logically equivalent to the statement that the sum of the interior angles of a triangle = 180°. Both of these reflect the underlying assumption that we are dealing with the geometry of a flat plane.

REVIEW OF BASIC CALCULUS

DIFFERENTIAL CALCULUS

Consider a function $f(x)$ of a single variable x. Then

$$\frac{df(x)}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \qquad (1)$$

In the world of physics it is often to view Δf and Δx as "a little bit of $f(x)$" or "a small change in x". In this spirit one can usefully write

$$\boxed{\frac{\Delta f}{\Delta x} = \frac{1}{\Delta x / \Delta f}} \qquad (2)$$

This relation will be useful in finding the derivatives of inverse functions. More later

Some Simple Examples:

$$f(x) = x^2 \Rightarrow \frac{f(x+\Delta) - f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} \qquad (3)$$

$$= \frac{\cancel{x^2} + 2x\Delta x + \overbrace{(\Delta x)^2}^{\approx 0} - \cancel{x^2}}{\Delta x} \cong \frac{2x\Delta x}{\Delta x} = 2x \qquad (4)$$

NOTE: It is implicit in (1)-(4) that $df(x)/dx$ leads to a unique limit for the function in question. When we discuss complex variables, we will see that not all functions lead to unique derivatives. Functions which do are said to be analytic.

Other Simple Examples:

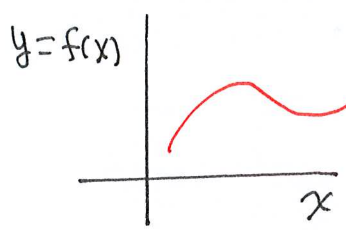
$$f(x) = x^3 \Rightarrow \frac{\Delta f(x)}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x} \qquad (5)$$

$$\frac{\Delta f(x)}{\Delta x} = \frac{\cancel{x^3} + 3x^2\Delta x + \overset{\to\,\approx\,0}{3x(\Delta x)^2 + (\Delta x)^3} - \cancel{x^3}}{\Delta x} \cong \frac{3x^2 \Delta x}{\Delta x} \qquad (6)$$

Hence: $$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = 3x^2 \qquad (7)$$

More generally, for any integer n: $$\boxed{\frac{d}{dx} x^n = n x^{n-1}} \qquad (8)$$

The same formula even holds for fractional powers. To simplify our notation we define $y \equiv f(x)$ which is useful when we imagine plotting the function $f(x)$:



Now consider the function $y = x^{1/n}$ where n = integer. Then

$$y = x^{1/n} \Rightarrow y^n = x \quad \text{or } x = y^n \quad n = \text{integer} \qquad (9)$$

Evidently, $$\frac{dx}{dy} = n y^{n-1} \Rightarrow \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{n y^{n-1}} \qquad (10)$$

Substituting $y = x^{1/n}$ in (10) $\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n} \cdot \frac{1}{x^{1-1/n}} \qquad (11)$$

$$\boxed{\frac{dy}{dx} = \left(\frac{1}{n}\right) x^{(1/n-1)}} \qquad (12)$$

As expected, this is the same formula as in (8) with n replaced by $1/n$. For $x^{1/2} = y \Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$ (12)

DERIVATIVES OF PRODUCTS & QUOTIENTS

Let $f(x)$ and $g(x)$ be two functions of x, each of which is assumed to be differentiable. Then from the definition of the derivative we have ⊛

$$D\,[f(x)\cdot g(x)] = f\,Dg(x) + g\,Df(x) \qquad (1)$$

In QM, and other advanced environments, $f(x)$ and $g(x)$ may be matrices which depend on x, or other objects for which it is the case that

$$f(x)\cdot g(x) \neq g(x)\cdot f(x) \qquad (2)$$

Equivalently,

$$\underbrace{[f(x), g(x)]}_{\text{"Commutator of } f \text{ and } g\text{"}} = f(x)\cdot g(x) - g(x)\cdot f(x) \neq 0 \qquad (3)$$

In such a case we must be careful to preserve the order of the operators/objects in (1), so that we should really write

$$\boxed{D\,[f(x)\cdot g(x)] = f(x)\,Dg(x) + Df(x)\cdot g(x)} \quad (4)$$

An example of non-commuting operators is

$$\boxed{[X, P] = i\hbar} \qquad (5)$$

X = operator which measures the x-coordinate of a particle
P = operator " " " x-component of the momentum

Equation (5) eventually leads to the HEISENBERG UNCERTAINTY PRINCIPLE, $\Delta x\,\Delta p_x \gtrsim \hbar$.

⊛ The rule in (1) & (4) can be derived as follows:

$$D\,[f(x)\cdot g(x)] \cong \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\cdot g(x)}{\Delta x}\,; \text{ as } \Delta x \to 0 \;\; f(x+\Delta x) \cong f(x) + \Delta x\, f'(x)$$

$$= \frac{\cancel{f(x)g(x)} + \Delta x\, f'(x)\cdot g(x) + \Delta x\, f(x)\, g'(x) - \cancel{f(x)g(x)}}{\Delta x} = \Delta x\,\{f'(x)g(x) + f(x)g'(x)\}\frac{1}{\Delta x}$$

$$\to f'(x)g(x) + g'(x)f(x) \;\checkmark$$

In the same manner the derivative of a quotient is given by

$$D(f/g) = \frac{gDf - fDg}{g^2}$$

<u>Mnemonic:</u> $D\left(\frac{hi}{ho}\right) = \frac{ho\,D\,hi - hi\,D\,ho}{ho \cdot ho}$

<u>Example:</u> $D\tan x = D\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot D\sin x - \sin x \cdot D\cos x}{\cos^2 x}$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

THE CHAIN RULE FOR DERIVATIVES

Suppose that we have a function that is a function of another function. For example $f = f(u(x)) = (x^2+1)^2$. (1)

This can be differentiated directly by writing

$$f = f(u(x)) = (x^2+1)^2 = x^4+2x^2+1 \Rightarrow \frac{df}{dx} = 4x^3+4x \quad ✓ \qquad (2)$$

But sometimes it is easier to write this as

$$f = f(u) = u^2, \quad u = x^2+1 \qquad (3)$$

Then the chain rule reads $\boxed{\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}}$ (4)

This makes sense if we imagine that du = "a little bit of u"

so that we can imagine "cancelling" du in Eq. (4).

Using (3) & (4) we then have $\frac{df}{dx} = \underbrace{(2u)}_{df/du}\cdot 2x = 2(x^2+1)\cdot 2x$ (5)

$$= 4x^3+4x \quad ✓$$

So we obtain the same result.

EXAMPLE 2: (AND A REVIEW OF SOME TRIG IDENTITIES!)

$$f(x) = \sin^2 x \;;\; \text{let } u = \sin x \text{ and } f(u) = u^2 \qquad (6)$$

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = (2u)\frac{du}{dx} = 2\sin x\cdot\frac{d}{dx}\sin x \qquad (7)$$

Let us anticipate a result we will prove later:

$$\boxed{\frac{d}{dx}\sin x = \cos x\;;\quad \frac{d}{dx}\cos x = -\sin x} \qquad (8)$$

Combining (7) & (8) we find:

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = 2\sin x\cdot\underbrace{\frac{d}{dx}\sin x}_{\cos x} = \underline{\underline{2\sin x\cdot\cos x}} \quad (9)$$

Let us do the same problem another way (good check!!)

Recall

$$\sin(x+y) = \sin x\cdot\cos y + \cos x\cdot\sin y \quad (10)$$

$$x=y \Rightarrow \quad \sin(2x) = \sin x\cdot\cos x + \cos x\sin x = 2\sin x\cdot\cos x \quad (11)$$

$$\cos(x+y) = \cos x\cdot\cos y - \sin x\cdot\sin y \quad (12)$$

$$x=y \Rightarrow \quad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \quad (13)$$

$$(13) \Rightarrow \sin^2 x = \frac{1-\cos(2x)}{2} \Rightarrow \frac{d}{dx}\sin^2 x = \frac{d}{dx}\left[\frac{1-\cos(2x)}{2}\right] \quad (14)$$

$$\Rightarrow \frac{d}{dx}\sin^2 x = -\frac{1}{2}\frac{d}{dx}\cos(2x)$$

$$\text{Now let } u = 2x \Rightarrow \frac{d}{dx}\cos(2x) = \frac{d\cos(u)}{du}\cdot\frac{du}{dx} = -\sin(u)\cdot 2 \quad (15)$$

$$= -\sin(2x)\cdot 2$$

$$\Rightarrow \frac{d}{dx}\sin^2 x = \left(-\frac{1}{2}\right)(-2\sin(2x)) = \sin(2x) \overset{(11)}{=} \underline{\underline{2\sin x\cdot\cos x}} \quad (16)$$

So we have obtained the same result as in (9)

EXAMPLE 3: $\quad f(x) = x^4 \Rightarrow \frac{df}{dx} = 4x^3 \quad (17)$

But what if we write

$$f = f(u) = u^2;\quad u = x^2 \Rightarrow \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = 2u\cdot 2x$$

$$= 2x^2\cdot 2x = 4x^3$$

CONCLUSION: There are many ways to find $\frac{df}{dx}$ (18)

Higher-Order Chain Rule:

The idea behind the chain rule can be extended beyond the simple cases we have done thusfar:

Example 4: Consider $f(x) = \sqrt{1-\cos^2 x}$ (1)

Here we can write immediately: $\sqrt{1-\cos^2 x} = \sin x \Rightarrow$ (2)

$$\frac{df}{dx} = \frac{d}{dx}\sin x = \cos x \checkmark \quad (3)$$

However, this can also be evaluated via the chain rule:

$$f = u^{1/2} \; ; \quad u = 1 - v^2 \; ; \quad v = \cos x \quad (4)$$

Working backwards from right to left we see that (4) reproduces f(x) in (1). The chain rule argument is now:

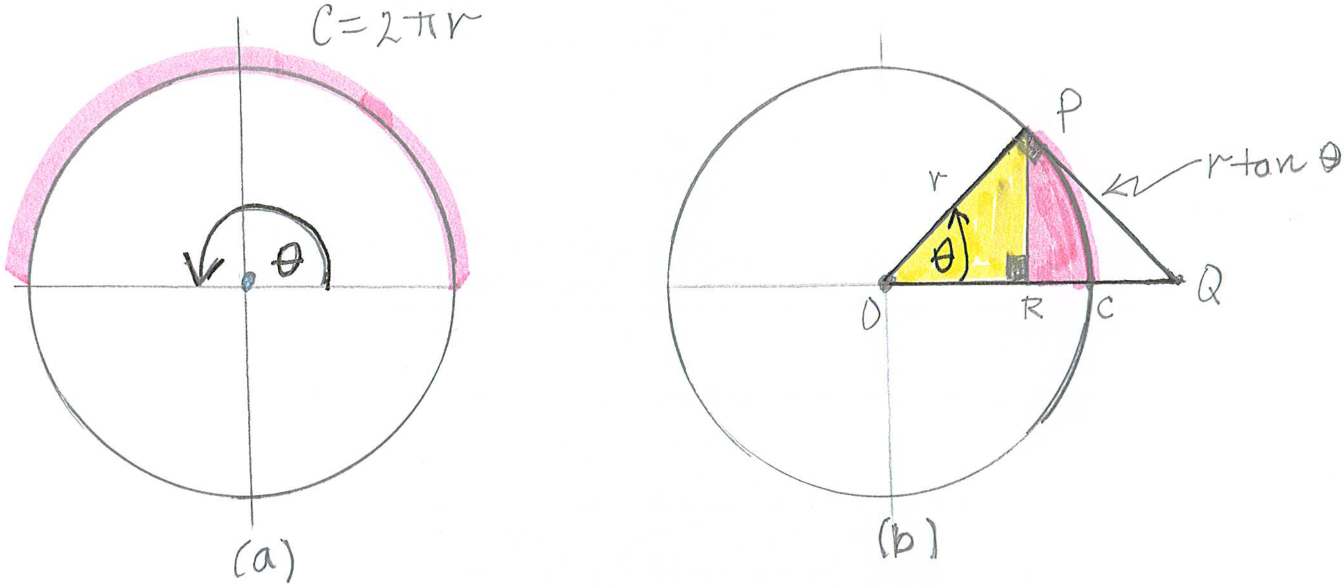
$$\frac{df(x)}{dx} = \underbrace{\frac{df}{du}}_{\frac{1}{2}u^{-1/2}} \cdot \underbrace{\frac{du}{dv}}_{-2v} \cdot \underbrace{\frac{dv}{dx}}_{-\sin x} = u^{-1/2} \cdot v \cdot \sin x \quad (5)$$

$$= \frac{1}{\sqrt{1-v^2}} \cdot \cos x \cdot \sin x \quad (6)$$

$$\Rightarrow \frac{df(x)}{dx} = \cancel{\frac{1}{\sqrt{1-\cos^2 x}}} \cdot \cos x \cdot \cancel{\sin x} = \cos x \checkmark \quad (7)$$

In complicated cases where the answer cannot be seen directly multiple applications of the chain rule may be useful.

UNDERSTANDING RADIANS



We want a measure of angles (θ) which relates θ to the fraction of the total circumfrence subtended by θ. From Figure (a) it is clear that the fraction of the cirumfrence that is shaded is directly proportional to θ so that

$$\frac{1}{2}C \Leftrightarrow 180° \quad \text{for any radius } r$$

$$\hookrightarrow \frac{1}{2}C_1 = \pi r_1 \qquad \frac{1}{2}C_2 = \pi r_2 \qquad (1)$$

Hence for any r $\boxed{180° \equiv \pi \text{ radians} \; ; \; 90° = \frac{\pi}{2} \text{ radians}}$

Hence 1 radian $\Leftrightarrow \dfrac{180°}{\pi} = 57.2957795°\ldots$

Because radians provide a direct measure of the length s of a circular arc, radians are a natural measure of θ:

$$\boxed{s = \theta \cdot r \quad (\theta \text{ in radians})} \quad (2)$$

RADIANS: IMPLICATIONS & APPLICATIONS

Many formulas involving $\sin\theta$, $\cos\theta$, $\tan\theta$, ... and other trig functions simplify if θ is expressed in radians. Consider the 3 regions in Figure (6) denoted by POR, POC, and POQ:

Evidently,

$$\text{area}(POR) \leq \text{area}(POC) \leq \text{area}(POQ) \qquad (1)$$

$$\text{area}(POR) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(PR)(OR) = \frac{1}{2}(r\sin\theta)(r\cos\theta)$$

$$= \frac{1}{2}r^2\sin\theta\cdot\cos\theta \qquad (2)$$

Note that this result holds however θ is measured.

Similarly,

$$\text{area}(POQ) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}r\cdot\underbrace{r\tan\theta}_{PQ}$$

$$= \frac{1}{2}r^2\tan\theta \qquad (3)$$

This result also holds however θ is measured.

Finally,

$$\text{area}(POC) = \underbrace{\left(\frac{\theta}{2\pi}\right)}_{\text{fraction of area of circle}^*}\cdot\underbrace{\pi r^2}_{\text{total area of circle}} \qquad (4)$$

* NOTE!! This formula holds only when θ is expressed in terms of radians!! Hence the following results hold only for θ in radians.

Combining the previous results we have

$$\text{area}(POR) \le \text{area}(POC) \le \text{area}(POQ)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\cancel{\tfrac{1}{2}}\, r^2 \sin\theta \cdot \cos\theta \le \cancel{\tfrac{1}{2}}\, \theta \cdot \cancel{r^2} \le \cancel{\tfrac{1}{2}}\, r^2 \tan\theta \Leftrightarrow \sin\theta/\cos\theta$$

dividing by $\sin\theta$:

$$\boxed{\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}} \qquad (5)$$

This is an exact result when θ is expressed in radians.

Next consider the limit as $\theta \to 0$, so that $\cos\theta \to 1$. Then

Eq. (5) $\Rightarrow$

$$\boxed{\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1} \qquad (6)$$

This means that when $\sin\theta$ is expanded in an infinite series in θ (when θ is expressed in radians) that the series must start as

$$\boxed{\sin\theta \cong \theta} \qquad \left(\begin{matrix}\theta \text{ in radians} \\ \text{and small}\end{matrix}\right) \qquad (7)$$

Similarly, for small θ

$$\boxed{\cos\theta = \sqrt{1 - \sin^2\theta} \cong \sqrt{1 - \theta^2} \cong 1 - \tfrac{1}{2}\theta^2} \qquad (8)$$

These formulas are very widely used and should be memorized. The higher order terms are given by

$$\sin\theta \cong \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \dots \qquad (9)$$

$$\cos\theta \cong 1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \dots$$

L'Hôpital's Rule

The result $\theta/\sin\theta \xrightarrow{\theta \to 0} 1$ provides an interesting example of this very useful rule, which is easy to understand. Suppose we want to evaluate

$$\lim_{x \to 0} \frac{f(x)}{g(x)} \quad \text{where } f(0)=0 \text{ and } g(0)=0 \qquad (1)$$

Then formally this rule states that

$$\boxed{\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(x=0)}{g'(x=0)}} \quad (2) \qquad f'(x) \equiv \frac{df(x)}{dx}, \text{ etc.}$$

<u>Example 1:</u> $\lim_{x \to 0} \frac{\sin x}{x}$ looks like $\lim_{x \to 0} \frac{0}{0}$? (3)

But the rule $\Rightarrow$ $\lim_{x \to 0} \frac{\sin x}{x} = \frac{\frac{d}{dx}(\sin x)\big|_{x=0}}{\frac{d}{dx}(x)\big|_{x=0}} = \frac{\cos x|_{x=0}}{1|_{x=0}} = 1$ ✓ (4)

<u>This recovers the previous result.</u>

In the event that $\frac{f'(x=0)}{g'(x=0)} = \frac{0}{0}$ the process can be continued so that,

$$\boxed{\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f''(x=0)}{g''(x=0)}} \quad (5)$$

Example 2:

There is a set of numbers known as the BERNOULLI NUMBERS, which are useful in evaluating integrals. $\rightarrow B_n (n = 0, 1, 2, \ldots)$. For our purposes:

$$B_0 = \lim_{x \to 0} \frac{x}{e^x - 1} \rightarrow \frac{0}{0} \quad \Big\} \text{ recall: } (\text{any number})^0 = 1 \qquad (6)$$

Applying L'Hôpital's Rule:

$$B_0 = \lim_{x \to 0} \frac{\frac{d}{dx} \cdot x}{\frac{d}{dx}(e^x - 1)} = \frac{1}{e^x} = \frac{1}{1} = 1 \qquad (7)$$

$$\Rightarrow \boxed{B_0 = 1}$$

Consider next:

$$B_1 \equiv \frac{d}{dx}\left(\frac{x}{e^x - 1}\right)_{x=0} \equiv \lim_{x \to 0} \frac{d}{dx}\left(\frac{x}{e^x - 1}\right) \qquad (8)$$

Using the quotient rule:

$$B_1 = \frac{d}{dx}\left(\frac{x}{e^x - 1}\right) = \frac{[(e^x - 1)\cdot 1 - x e^x]}{(e^x - 1)^2} \xrightarrow{x \to 0} \frac{0}{0} \qquad (9)$$

Applying L'Hôpital's Rule:

$$B_1 = \frac{\frac{d}{dx}[(e^x - 1)1 - x e^x]}{\frac{d}{dx}(e^x - 1)^2} = \frac{\cancel{e^x} - x e^x - \cancel{e^x}}{2(e^x - 1)} = \frac{0}{0} \qquad (10)$$

Applying L'Hôpital's Rule again:

$$\boxed{B_1 = \frac{\frac{d}{dx}(-x e^x)}{\frac{d}{dx}(2)(e^x - 1)} = \frac{[-x e^x - e^x]_{x=0}}{[2 e^x]_{x=0}} = \frac{-1}{2}} \qquad (11)$$

INTRODUCTION TO π

References: [1] "A History of π" - Peter Beckmann
(Plenum, New York, 1971)

[2] "The Joy of π" - David Blatner (Walker, 1997) *

[3] "π Unleashed - Jörg Arndt & Christoph Haenel

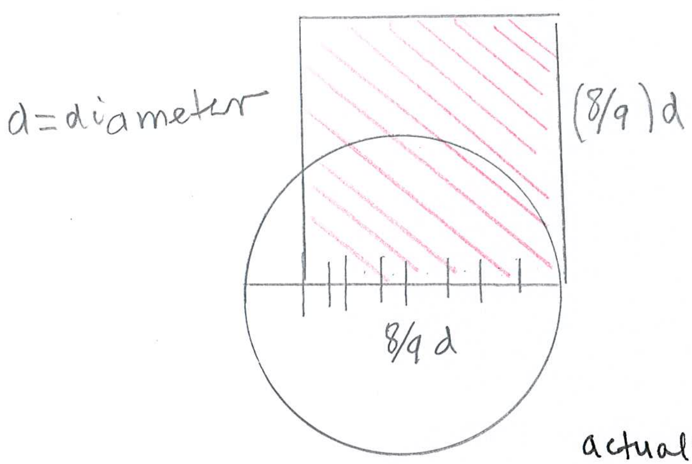
[4] "A Study on the Randomness of the Digits of π,"
S.-J. Tu and E. Fischbach, Int. J. Mod. Phys. C16, 281 (2005)

First Determination of π: (Ahmes * ~1650 BCE, Egypt

$\Downarrow$

area of square $= (8/9)^2 d^2$

$= \text{area of circle} = \dfrac{\pi d^2}{4}$



This gives:

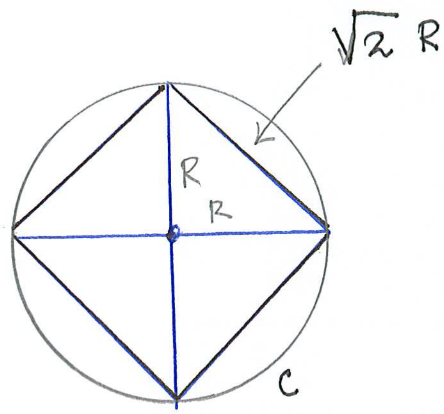
$$\frac{\pi}{4} = \frac{64}{81} \Rightarrow \pi = \frac{256}{81} = 3.16049$$

actual: $\pi = 3.14159265\ldots$

Hence the Ahmes value was good to $\dfrac{0.0189}{3.14159} \cong 0.6\%$

It is not exactly clear how Ahmes came to this amazing result!!

Method of Exhaustion:

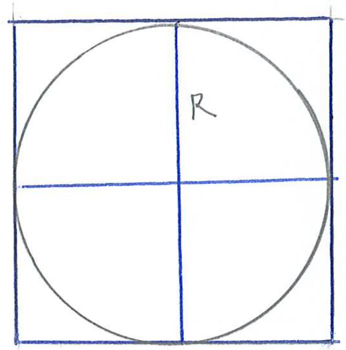


Circle: $C = P = 2\pi R \cong 6.28R$

square: $P = 4\sqrt{2}\,R = 5.66R$

$\dfrac{\text{circle}}{\text{square}} \Rightarrow \pi \cong 2.83$

↳ inscribed



Circle: $C = P = 2\pi R$

Square: $P' = 4 \times 2R = 8R$

$\dfrac{\text{circle}}{\text{square}} \Rightarrow \pi = 4$

↳ circumscribed

Taking the average of these two results gives

$\pi \cong \frac{1}{2}(2.83 + 4) = 3.41$ (good to ~ 8%)

In the "method of exhaustion" the squares were continually subdivided, so that their perimeters (and/or areas) more closely approximated those of the circle.

In some sense this can be thought of as the origins of calculus.

10

23 וַיַּעַשׂ אֶת־הַיָּם מוּצָק עֶשֶׂר
בָּאַמָּה מִשְּׂפָתוֹ עַד־שְׂפָתוֹ עָגֹל ׀ סָבִיב וְחָמֵשׁ בָּאַמָּה
קוֹמָתוֹ וְקָוֹ שְׁלֹשִׁים בָּאַמָּה יָסֹב אֹתוֹ סָבִיב׃

30

23. και εποιησε την θαλασσαν δεκα εν πηχει απο του χειλους αυτης εως του ψειλους αυτης, στρογγυλον κυκλω το αυτο. πεντε εν πηχει το υψος αυτης. και συνηγμενη τρεις και τριακοντα εν πιχει.

[23] Hizo asimismo un mar de fundición, de diez codos del uno al otro lado, redondo, y de cinco codos de alto, y ceñialo en derredor un cordón de treinta codos.

23. Il fit aussi une mer de fonte, de dix coudées d'un bord jusqu'à l'autre, qui était toute ronde: elle avait cinq coudées de haut, et elle était environnée tout à l'entour d'un cordon de trente coudées.

23. Udělal též moře slité, desíti loket od jednoho kraje k druhému, okrouhlé vůkol, a pět loket byla vysokost jeho, a okolek jeho třicet loket vůkol.

23. Und er machte ein Meer, gegossen, von einem Rand zum andern zehn Ellen weit, rundumher, und fünf Ellen hoch, und eine Schnur dreißig Ellen lang war das Maß ringsum.

23. And he made a molten sea, ten cubits from the one brim to the other; it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.

length EA and mark it on a piece of rope. Then we straighten the rope and lay it off along AB as many times as it will go. It will go into our unit distance AB between 7 and 8 times. (Actually, if we swindle a little and check by 20th century arithmetic, we find that 7 is much nearer the right value than 8, i.e., that E_7 in the figure on p. 13 is nearer to B than E_8, for $1/7 = 0.142857\ldots$, $1/8 = 0.125$, and the former value is nearer $\pi - 3 = 0.141592\ldots$ However, that would be difficult to ascertain by our measurement using thick, elastic ropes with coarse charcoal marks for the roughly circular curve in the sand whose surface was judged "flat" by arbitrary opinion.)

We have thus measured the length of the arc EA to be between 1/7 and 1/8 of the unit distance AB; and our second approximation is therefore

$$3\tfrac{1}{8} < \pi < 3\tfrac{1}{7} \qquad (4)$$

for this, to the nearest simple fractions, is how often the unit rope length AB goes into the circumference $ABCD$.

And indeed, the values

$$\pi = 3, \qquad \pi = 3\tfrac{1}{7}, \qquad \pi = 3\tfrac{1}{8}$$

are the values most often met in antiquity.

For example, in the Old Testament (I Kings vii.23, and 2 Chronicles iv.2), we find the following verse:

> *"Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about."*

The molten sea, we are told, is round; it measures 30 cubits round about (in circumference) and 10 cubits from brim to brim (in diameter); thus the biblical value of π is $30/10 = 3$.

The Book of Kings was edited by the ancient Jews as a religious work about 550 B.C., but its sources date back several centuries. At that time, π was already known to a considerably better accuracy, but evidently not to the editors of the Bible. The Jewish Talmud, which is essentially a commentary on the Old Testament, was published about 500 A.D. Even at this late date it also states "that which in circumference is three hands broad is one hand broad."

FROM: "A HISTORY OF π" by Petr Beckmann

Every country has its circle-squarers, but the following will be limited to the circle-squarers of America.

There is a story about some American legislature having considered a bill to legislate, for religious reasons, the biblical value of $\pi = 3$. I have found no confirmation of this story; very probably it grew out of an episode that actually took place in the State Legislature of Indiana in 1897. The Indiana House of Representatives did consider and unanimously pass a bill that attempted to legislate the value of π (a wrong value); the author of the bill claimed to have squared the circle, and offered this contribution as a free gift for the sole use of the State of Indiana (the others would evidently have to pay royalties).

The author of the bill was a physician, Edwin J. Goodman, M.D., of Solitude, Posey County, Indiana, and it was introduced in the Indiana House on January 18, 1897, by Mr. Taylor I. Record, Representative from Posey County. It was entitled "A bill introducing a new Mathematical truth,"[89] and it became House Bill No. 246; copies of the bill are preserved in the Archives Division of the Indiana State Library; the full text has also been reprinted in an article by W.E. Eddington in 1935.[90]

The preamble to the bill informs us that this is

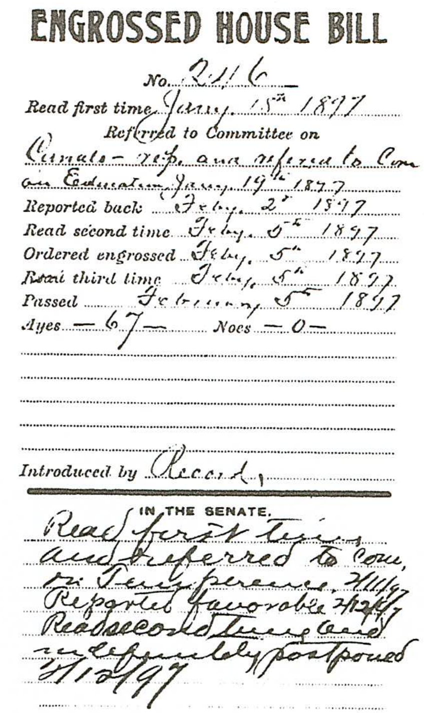
> *A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature in 1897.*

The bill consisted of three sections. Section 1 starts off like this:

> *Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle's area is entirely wrong . . .*

An "equilateral rectangle" is, of course, a square, so that the first statement does not make any sense at all; but if we give the author the benefit of the doubt and assume that this is a transcript error for "equilateral triangle," then what Mr. Goodwin of Solitude, Posey County, had discovered in his first statement was the equivalent of $\pi = 16/\sqrt{3} = 9.2376\ldots$, which probably represents the biggest overestimate of π in the history of mathematics.

(9)


ENGROSSED HOUSE BILL

No. 246

Read first time Jany. 15th 1897

Referred to Committee on Canals – refd. and referred to Com on Education Jany. 19th 1897

Reported back Feby. 2nd 1897

Read second time Feby. 5th 1897

Ordered engrossed Feby. 5th 1897

Read third time Feby. 5th 1897

Passed February 5th 1897

Ayes — 67 — Noes — 0 —

Introduced by Record

IN THE SENATE.

Read first time and referred to Com. on Temperance, 2/11/97

Reported favorable 2/12/97

Read second time and indefinitely postponed 2/12/97

Facsimile of Bill No. 246, Indiana State Legislature, 1897. Kindly made available by the Indiana State Library.

However, Sections 1 and 2 contain more hair-raising statements which not only contradict elementary geometry, but also appear to contradict each other. Section 2 of the bill concludes

> *By taking the quadrant of the circle's circumference for the linear unit, we fulfill the requirements of both quadrature and rectification of the circle's circumference. Furthermore, it has revealed the ratio of the chord and arc of ninety degrees, which is as seven to eight, and also the ratio of the diagonal and one side of a square which is as ten to seven, disclosing the fourth important fact, that the ratio of the diameter and circumference is as five-fourths to four; and because of these facts and the further fact that the rule in present use fails to work both ways mathematically, it should be discarded as wholly wanting and misleading in practical applications.*

$$\frac{d}{c} = \frac{5/4}{4} \Rightarrow \text{"}\pi\text{"} = \frac{16}{5} = 3.2$$