From the basic identity in (7) we have (setting $l=j$)

$$\epsilon_{ijk}\,\epsilon_{kjm} = \underbrace{\delta_{ij}\,\delta_{jm}}_{\delta_{im}} - \delta_{jj}\,\delta_{im} \qquad (11)$$

We note that $\boxed{\delta_{jj} \equiv \sum_j \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3}$. Hence (12)

$$\boxed{\epsilon_{ijk}\,\epsilon_{kjm} = \delta_{im} - 3\,\delta_{im} = -2\,\delta_{im}} \qquad (13)$$

Finally, setting $m=i$ in (13) gives $\boxed{\epsilon_{ijk}\,\epsilon_{kji} = -2\delta_{ii} = -6}$ (14)

Collecting these results together, and rearranging some indices, we get

$$\boxed{\begin{aligned} \epsilon_{ijk}\,\epsilon_{klm} &= \delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im} \\ \epsilon_{ijk}\,\epsilon_{kmj} &= 2\,\delta_{im} \\ \epsilon_{ijk}\,\epsilon_{ijk} &= 6 \end{aligned}} \qquad (15)$$

Applications:

[1] $\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i\,(\vec{B}\times\vec{C})_i \overset{(5)}{=} A_i\,(\epsilon_{ijk}\,B_j C_k) = \epsilon_{ijk}\,A_i B_j C_k$

$$= \det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \qquad (16)$$

Using the antisymmetry of ϵ_{ijk} we can write:

$$\vec{A}\cdot\vec{B}\times\vec{C} = \epsilon_{ijk}\,A_i B_j C_k = +\epsilon_{kij}\,C_k A_i B_j = \vec{C}\cdot\vec{A}\times\vec{B} \qquad (17)$$

$$\therefore\ \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \qquad (18)$$

$$\text{also} \quad = \vec{B}\cdot\vec{C}\times\vec{A}$$

[2] Consider next simplifying $\vec{A}\times(\vec{B}\times\vec{C})$:

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk} A_j \underbrace{(\vec{B}\times\vec{C})_k}_{\epsilon_{k\ell m} B_\ell C_m} \tag{19}$$

Hence $[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk}\epsilon_{k\ell m} A_j B_\ell C_m = (\delta_{i\ell}\delta_{jm} - \delta_{\ell j}\delta_{im}) A_j B_\ell C_m$ (20)

$$= B_i(\vec{A}\cdot\vec{C}) - C_i(\vec{A}\cdot\vec{B}) \Leftrightarrow \boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})} \tag{21}$$

"BAC" – "CAB"

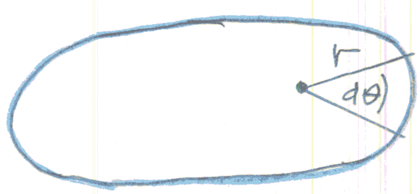
[3] Differentiation of Cross Products

$$\vec{L} = \vec{r}\times\vec{p} \Rightarrow \frac{d\vec{L}}{dt} = \underbrace{\frac{d\vec{r}}{dt}\times\vec{p}}_{\vec{v}\times m\vec{v}=0} + \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\vec{F} \tag{22}$$

For a central force $\vec{r}\parallel\vec{F} \Rightarrow d\vec{L}/dt = 0$ (23)

This is Kepler's 2ND Law, usually stated as the constancy of the areal velocity):

$$dA = \tfrac{1}{2}(r\,d\theta)r = \tfrac{1}{2}r^2 d\theta \tag{24}$$



$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega \tag{25}$$

$$L = mvr = m\omega r^2 \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{constant} \tag{26}$$

Hence the constancy of the areal velocity is a consequence of the constancy of the (orbital) angular-momentum L.

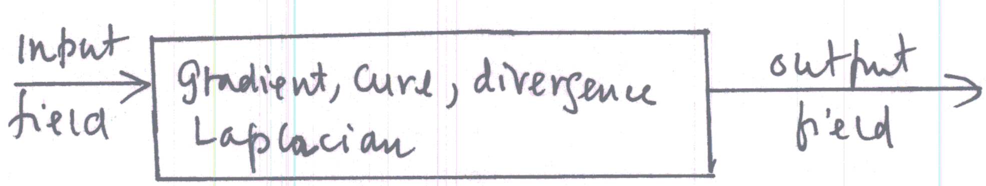
SCALAR, VECTOR & TENSOR FIELDS

[1] A <u>scalar</u> field is a function $\phi(\vec{r})$ which assigns to each point $\vec{r}$ in space a scalar ϕ. [Examples: temperature distribution, density distribution, electromagnetic scalar potential.]

[2] A <u>vector</u> field $\vec{A}(\vec{r})$ assigns to each point a vector $\vec{A}$. [Examples: electromagnetic fields $\vec{E}(\vec{r})$, $\vec{B}(\vec{r})$; gravitational field $\vec{g}(\vec{r})$; velocity field in a fluid $\vec{v}(\vec{r})$]

[3] A <u>tensor</u> field $g_{ij}(\vec{r})$ assigns to each point in space a tensor (g_{ij}) quantity. [Examples: metric tensor g_{ij}, energy-momentum tensor T_{ij}, electromagnetic field tensor $F_{\mu\nu}$]

The familiar differential operators act on these fields and produce other fields:



PHYSICAL INTERPRETATION OF DIV, GRAD, CURL, ...

See: H. M. Schey, *Div, Grad, Curl, and All That* (Norton, New York, 1973)

Gradient: Given a scalar function $u = u(\vec{r}) = u(x,y,z)$ we define

$$\text{grad}\, u \equiv \vec{\nabla} u = \hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z} = \text{VECTOR} \qquad (1)$$

One can develop a physical picture of $\vec{\nabla} u$ can be obtained by noting that the scalar change du of u is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \underbrace{\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}\right)}_{\vec{\nabla} u} \cdot \underbrace{(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz)}_{d\vec{r}} \qquad (2)$$

Hence

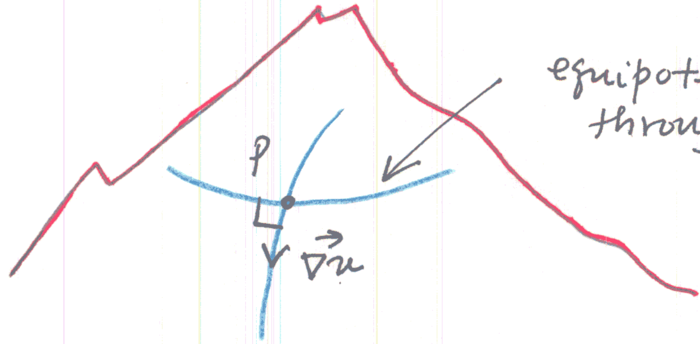
$$\boxed{du = \vec{\nabla} u \cdot d\vec{r} = |\vec{\nabla} u|\,|d\vec{r}|\cos\theta} \qquad (3)$$

$\cos\theta$ is the angle between $\vec{\nabla} u$ and $d\vec{r}$. We see that du is a maximum when $\vec{\nabla} u$ and $d\vec{r}$ point in the same direction, so that

$$|\vec{\nabla} u| = |du| / |d\vec{r}| \qquad (4)$$

Summary: If we move from $\vec{r}$ to $(\vec{r} + d\vec{r})$, then $u(\vec{r})$ will change by an amount du. From (3) we see that this change will be a maximum when $d\vec{r}$ is chosen to be in the direction of $\vec{\nabla} u$. So $\vec{\nabla} u$ points in the direction in which $u(\vec{r})$ increases most rapidly. Hence $\vec{\nabla} u$ extracts from $u(\vec{r})$ the information about the direction in which $u(\vec{r})$ is changing most rapidly.

Example:



Consider a mountain where we characterize each point by a gravitational potential energy $u(P = x, y, z)$. Then $\vec{\nabla}u(P)$ points along the fall line which is the path of steepest descent. If a ski came loose, this is the path it would take.

Divergence

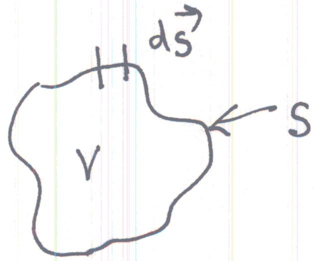
: This acts on Vector fields to produce a Scalar

$$\text{div}\,\vec{A} \equiv \vec{\nabla}\cdot\vec{A} \equiv \vec{\partial}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \equiv \partial_i A_i = \underline{\text{SCALAR}}$$

One important identity involving $\vec{\nabla}\cdot\vec{A}$ is GAUSS' THEOREM:

$$\int_V \vec{\nabla}\cdot\vec{A}\, dV = \int_S \hat{n}\cdot\vec{A}\, dS = \int_S \vec{A}\cdot(\hat{n}\, dS) = \int_S \vec{A}\cdot d\vec{S}$$



CURL : This operator acts on vector fields and produces another vector field.

Define $\partial_x \equiv \partial/\partial x$ etc.

$$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \vec{\partial}\times\vec{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x) \quad (1)$$

determinant

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \Rightarrow \boxed{(\nabla\times A)_i = \sum_{j,k} \epsilon_{ijk}\,\partial_j A_k} \quad (2)$$

The last representation is useful in proving certain identities such as:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \partial_i(\vec{\nabla}\times\vec{A})_i = \partial_i(\epsilon_{ijk}\,\partial_j A_k) \quad (3)$$

$$= \underbrace{\epsilon_{ijk}}_{\text{antisymmetric in } (i\Leftrightarrow j)}\;\underbrace{\partial_i\partial_j}_{\text{symmetric in } (i\Leftrightarrow j)} A_k \equiv 0 \quad (4)\ (5)$$

To show this is zero: $\epsilon_{ijk}\,\partial_i\partial_j \overset{(i\Leftrightarrow j)}{=} +\epsilon_{jik}\,\partial_j\partial_i = -\epsilon_{ijk}\,\partial_j\partial_i$ (6)

$$= -\epsilon_{ijk}\,\partial_i\partial_j = 0$$

Alternatively, do this by components: $\epsilon_{ijk}\,\partial_i\partial_j \to \epsilon_{ij3}\,\partial_i\partial_j$ (7)

$$= \epsilon_{123}\,\partial_1\partial_2 + \epsilon_{213}\,\partial_2\partial_1 = +\partial_1\partial_2 - \partial_2\partial_1 = 0 \quad \text{etc.}$$

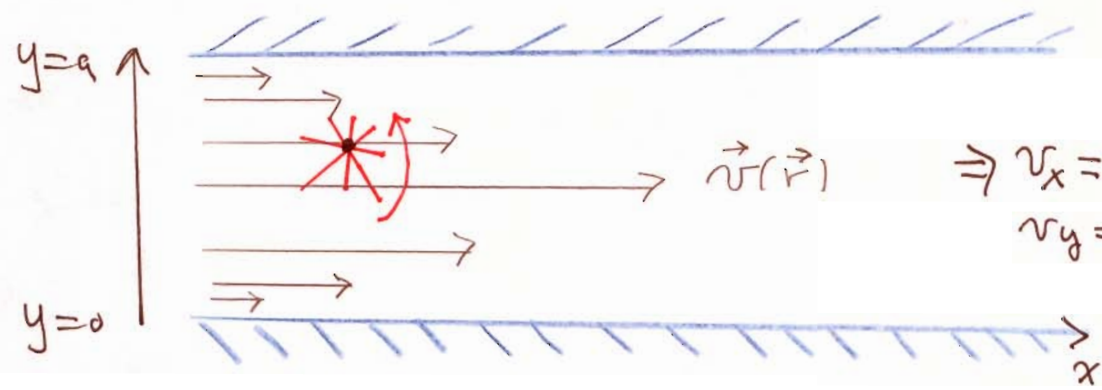
Hence returning to (3) we have shown that $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0}$ (8)

It follows that any field (such as the magnetic field $\vec{B}$) which can be expressed as the curl of another field ($\vec{B} = \vec{\nabla}\times\vec{A}$) has zero divergence.

Physical Interpretation of Curl:

Given a vector field [such as the velocity $\vec{v}(\vec{r})$ in the example below], that field will have a non-vanishing curl at $\vec{r}$ if a minute "paddle-wheel" placed at $\vec{r}$ will rotate. This can happen even if all the field lines for $\vec{v}(\vec{r})$ are straight:

As an example consider the flow of a river:



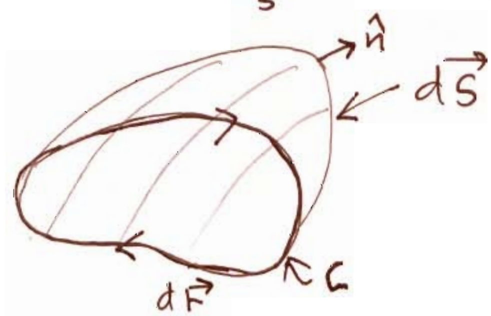
$$\Rightarrow v_x = v_0\, y(y-a)$$
$$v_y = 0$$

In this example $dv_x/dy = (2y-a)v_0 \Rightarrow v_x$ is a ~~maximum~~ minimum at $a/2$ ✓

It then follows that $\boxed{(\vec{\nabla}\times\vec{v})_z = \hat{z}\,(\partial_x v_y - \partial_y v_x) = -\hat{z}\, v_0\,(2y-a) \neq 0.}$

Hence there is in general a non-zero curl, even though the field lines are straight. Note that $(\vec{\nabla}\times\vec{v})_z = 0$ at $y = a/2$ as expected on symmetry grounds.

Stokes' Theorem: $$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,dS = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{r}$$

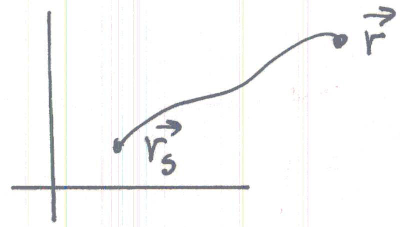


CONSERVATIVE FIELDS:

One can gain additional insight into the meaning of the curl by noting that the condition for the existence of a CONSERVATIVE FIELD $\vec{F}(\vec{r})$ is

$$\boxed{\vec{\nabla}\times\vec{F}(\vec{r})=0} \qquad (1)$$

Proof:



We define the work W done by a force $\vec{F}(\vec{r})$ along any path as

$$W=\int_{\vec{r}_1}^{\vec{r}_2}\vec{F}\cdot d\vec{r} \qquad (2)$$

The potential $V(\vec{r})$ can then be define as $\boxed{V(\vec{r})=-\int_{\vec{r}_s}^{\vec{r}}\vec{F}(\vec{r}\,')\cdot d\vec{r}\,'}$ (3)
where $\vec{r}_s$ is some standard reference point.

Then (3) $\Rightarrow$ $\boxed{dV(\vec{r}) = -\vec{F}(\vec{r})\cdot d\vec{r}}$ (4)

But this is equivalent to writing $\boxed{\vec{F}(r)=-\vec{\nabla}V(\vec{r})}$ (5) Since (5) $\Rightarrow$

$$\vec{F}\cdot d\vec{r}=-\vec{\nabla}V\cdot d\vec{r}=-\left(\hat{x}\frac{\partial V}{\partial x}+\hat{y}\frac{\partial V}{\partial y}+\hat{z}\frac{\partial V}{\partial z}\right)\cdot\left(\hat{x}dx+\hat{y}dy+\hat{z}dz\right)$$

$$=-\left(\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz\right)\equiv -dV \qquad (6)$$

Hence altogether:

$$\boxed{V(r)=-\int_{r_s}^{r}\vec{F}(\vec{r}\,')\cdot d\vec{r}\,' \Leftrightarrow dV(\vec{r})=-\vec{F}\cdot d\vec{r} \Leftrightarrow \vec{F}(\vec{r})=-\vec{\nabla}V(\vec{r})} \qquad (7)$$

Suppose now that we have a field $\vec{F}(\vec{r})$ that can be represented as the gradient of some potential $V(\vec{r})$ as in (7). It is straightforward to show that such a field $\vec{F}$ satisfies

$$\vec{\nabla}\times\vec{F}=0$$

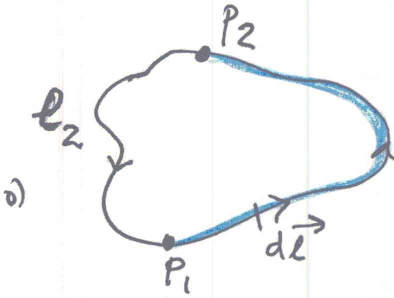
$$(\vec{\nabla}\times\vec{F})_i = \epsilon_{ijk}\,\partial_j F_k \qquad F_k = -\partial_k V \quad \text{This is the same as } \vec{F}=-\vec{\nabla}V$$

$$(\vec{\nabla}\times\vec{F})_i = -\underbrace{\epsilon_{ijk}\,\partial_j\partial_k}_{=0} V \equiv 0 \quad \leftarrow \text{using } 17(4) \tag{8}$$

Hence $$\boxed{\vec{F}=-\vec{\nabla}V \Rightarrow \vec{\nabla}\times\vec{F}=0} \tag{9}$$

We next show that the implication goes the other way too:

Using Stokes' theorem [p.18]



$$\underbrace{\int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}}_{0 \text{ from } (9)} = \oint_C \vec{F}\cdot d\vec{\ell} = 0 \tag{10}$$

Then $$0=\oint \vec{F}\cdot d\vec{\ell} = \underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{\ell}}_{\text{along } \ell_1} + \underbrace{\int_{P_2}^{P_1} \vec{F}\cdot d\vec{\ell}}_{\text{along } \ell_2} \tag{11}$$

From (11) $$\underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{\ell}}_{\text{over } \ell_1} = -\underbrace{\int_{P_2}^{P_1} \vec{F}\cdot d\vec{\ell}}_{\text{over } \ell_2} = +\underbrace{\int_{1}^{P_2} \vec{F}\cdot d\vec{\ell}}_{\text{over } \ell_2} \tag{12}$$

Hence when $\vec{\nabla}\times\vec{F}=0$ the value of $\int \vec{F}\cdot d\vec{\ell}$ between any 2 points P_1 and P_2 is <u>independent of the path</u> (ℓ_1 or ℓ_2) between these points. But this $\Rightarrow$ $\boxed{\vec{F}\cdot d\vec{\ell} = -dV}$: (13)