THE EXPONENTIAL FUNCTION e^x

As with most mathematical functions there are many ways to introduce/derive e^x. For our purposes a simple way (among many!) is to define e^x by the series

$$e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (1)$$

Evidently the number $e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \qquad (2)$

$$\underbrace{1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots}_{2.666\ldots} = 2.7(1828)(1828)\ldots \qquad (3)$$

(a transcendental number) - like π

Why is this interesting? Consider the derivative of the series (1):

$$\frac{d}{dx} e^x = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \qquad (4)$$

$$= 1 + \frac{\cancel{2}x}{\cancel{2}\cdot 1} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2\cdot 1} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3\cdot 2\cdot 1} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Hence $\boxed{\frac{d}{dx} e^x = e^x}$ (5)

The function e^x can also be arrived at in other ways. Consider the function $f(x)$ defined by

$$f(x) = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N \qquad (6)$$

and use the chain rule: $\frac{d}{dx} f(u(x)) = \frac{df}{du} \cdot \frac{du}{dx}$ (7)

Let $u(x) = \left(1+\frac{x}{N}\right)$; $u^N = \left(1+\frac{x}{N}\right)^N$ (8)

Then $$\frac{df(x)}{dx} = \lim_{N\to\infty} \frac{d}{dx}(u^N) = \lim_{N\to\infty} (N u^{N-1}) \frac{du}{dx}$$ (9)

$\frac{du}{dx} \leftrightarrow \left(\frac{1}{N}\right)$

$$\frac{df(x)}{dx} = \lim_{N\to\infty} \left\{ (N u^{N-1})\left(\frac{1}{N}\right) = u^{N-1} = \frac{u^N}{u} \right\}$$ (10)

$$= \lim_{N\to\infty} \left\{ \frac{\left(1+\frac{x}{N}\right)^N}{1+\frac{x}{N}} \right\}$$ (11)

In the limit $N\to\infty$ we can approximate $1+\frac{x}{N} \approx 1$. Hence (11) $\Rightarrow$

$$\boxed{\frac{df(x)}{dx} = \lim_{N\to\infty} \left(1+\frac{x}{N}\right)^N = f(x)}$$ (12)

Comparing (12) & (5) we see that this is the same differential equation as for e^x. Hence it must be that $f(x) = e^x + \text{constant}$.

But $e^0 = 1$ and $f(0) = 1$ also $\Rightarrow$ $f(x) = e^x$ (13)

INTERPRETATION/APPLICATION OF (12) & (13):

Suppose you deposit d\$ into your bank account on January 1. Let x be the interest rate, for example $x = 8\% = 0.08$. If interest is evaluated at the end of the year the value of your account is

$$V(\text{Dec.}31) = \$d(1.08)$$ (14)

Suppose, however that the interest is credited twice/year at the rate of $\frac{1}{2}\times 0.08 = 0.04$ each time. Then the value of your account would now be:

$$V(\text{Dec. }31) = \$d(1.04)(1.04) = \$d(1+2\times 0.04+.0016)$$
$$= \$d(1.0816) \qquad (15)$$

So you are now slightly better off due to the more frequent compounding. Continuing, suppose interest is compounded quarterly. Then,

$$V(\text{Dec. }31) = \$d(1.02)(1.02)(1.02)(1.02) = \$d\,1.0824 \qquad (16)$$
$$= \$d\left(1+\frac{0.08}{4}\right)^4$$

So more frequent compounding helps!!

However, this does not go on forever. In the limit as compounding happens infinitely often then,

$x = 0.08$

$$V(\text{Dec. }31) = \$d \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = \$d\cdot e^x = \$d\cdot e^{0.08}$$
$$= \$d(1.08329)$$

"THE RULE OF 72"

This is a simple mnemonic to calculate how long it will take to double how much money you have, given an assumed constant rate of growth r (% per year).

At $t=0$ you have \$d. After 1 year you will then have \$d(1+r). After n years you will then have \$d$(1+r)^n$.

Hence we want to find n such that

$$\$d(1+r)^n = 2\$d \Rightarrow (1+r)^n = 2 \Rightarrow \boxed{n \ln(1+r) = \ln 2} \quad (1)$$

If we assume a conventionally modest value of $r \ll 1$, we can then expand,

$$\ln(1+r) \cong r \Rightarrow n \ln(1+r) \cong nr = \ln 2 \cong 0.693 \quad (2)$$

Hence

$$n = \frac{0.693}{r} = \frac{69.3}{100r} \cong \frac{72}{R} \quad R = \text{growth rate} \quad (3)$$

R = growth rate, i.e. $100 \times r$ (6% → 6)

Final result: $\boxed{n \cong \frac{72}{R} = \text{"rule of 72"}}$ (4)

Examples:	Rule (4)	Exact (1)
$r = 4\% \Rightarrow R=4$	$n=18$	17.7
$r = 6\% \Rightarrow R=6$	$n=12$	11.9
$r = 8\% \Rightarrow R=8$	$n=9$	9.006
$r = 9\% \Rightarrow R=9$	$n=8$	8.04
$r = 12\% \Rightarrow R=12$	$n=6$	6.11

APPLICATION TO RADIOACTIVE DECAY AND POPULATION GROWTH

Natural radioactivity was discovered in 1896 by Henri Becquerel. It was indicated by subsequent experiments that if one started with a sample of N_0 atoms at $t=0$ then the number remaining at a time $t > t=0$ is given by

$$N(t) = N_0 e^{-\lambda t} \Rightarrow \frac{dN(t)}{dt} = -\lambda \underbrace{N_0 e^{-\lambda t}}_{N(t)} = -\lambda N(t) \quad (1)$$

$$\Rightarrow \boxed{-\frac{dN(t)}{N(t)} = \lambda\, dt} \quad (2) \qquad \lambda \equiv \text{decay constant (units of } 1/\text{time})$$

Half Life $\equiv T_{1/2}$; this is the time at which $N(t)/N_0 = 1/2$.

From (1) & (2): $\underbrace{\frac{N(t)}{N_0}}_{1/2} = e^{-\lambda t} \Rightarrow \underbrace{\ln(1/2)}_{\ln(1) - \ln(2)} = -\lambda t$, $\ln(1) = 0$ $\Big\}$ $-\ln(2) = -\lambda t \equiv -\lambda T_{1/2}$

$$\Rightarrow \boxed{T_{1/2} = \frac{\ln 2}{\lambda} \cong \frac{0.693}{\lambda}} \quad (3)$$

- Half-lives of radioactive isotopes range from fractions of a second to tens of billions of years.
- There are recent indications that λ itself (and hence $T_{1/2}$) is not a constant. See "THROUGH THE WORMHOLE WITH MORGAN FREEMAN", July 13, 2011 story.
- For population growth Eq. (1) also applies with a change in sign: From (2) this gives

$$\boxed{\frac{dN(t)}{N(t)} = +\lambda\, dt} \quad (4)$$

INTRODUCTION TO $\ln(x)$

(natural logarithms of x)

As with other functions, $\ln(x) \equiv \ln x$ can be introduced in a number of ways. We choose to define $\ln x$ as the inverse of e^x as defined by

$$\boxed{x = e^{\ln x}} \qquad (1)$$

$$x \longrightarrow \boxed{\ln x} \longrightarrow \boxed{e^{\ln x}} \longrightarrow x$$

Then for $x = e \Rightarrow e = e^1 = e^{\ln e} \Rightarrow \ln e = 1$. We know how to differentiate e^x, and we can use this to differentiate $\ln x$:

Start with the chain rule $\quad \dfrac{d\, f(u(x))}{dx} = \dfrac{df}{du} \cdot \dfrac{du}{dx}$ "cancel" (2)

Then let $f(u) = e^u$; $u = \ln x$

From (1) & (2)

$$\frac{d}{dx}\{x\} = \frac{d}{dx}\{e^{\ln x}\}$$

$$1 = \underbrace{e^{\ln x}}_{x} \cdot \frac{d\ln x}{dx} = x \cdot \frac{d\ln x}{dx} \Rightarrow \boxed{\frac{d\ln x}{dx} = \frac{1}{x}} \qquad (3)$$

Continuing in this way we find:

$$\frac{d}{dx}\ln x \equiv D\ln x = \frac{1}{x}; \quad \frac{d^2}{dx^2}\ln x = \frac{-1}{x^2} \equiv D^2 \ln x; \quad D^3 \ln x = \frac{2}{x^3}, \ldots \qquad (4)$$

$$D^n \ln x = \frac{(-1)^{n+1}(n-1)!}{x^n} \qquad (5)$$

MORE ON LOGARITHMS ("logs")

From our previous definitions of ln a:

$$a = e^{\ln a} \qquad b = e^{\ln b} \qquad (1)$$

$$\Rightarrow \quad ab = e^{\ln a} \cdot e^{\ln b} = e^{\ln a + \ln b} \equiv e^{\ln(ab)} \qquad (2)$$

But, whatever ab is, it is just another number so that

Hence it must be that $\boxed{\ln(ab) = \ln(a) + \ln(b)}$ (3)

this is how a slide rule works!

From (1) we can also write : $\boxed{a^x = (e^{\ln a})^x = e^{x \ln a}}$ (4)

This will be important later in the semester when we discuss complex variables: If $z = x + iy$ we can determine the properties of e^z and $\ln z$. Using (4) we can then write

$$\boxed{z_1^{z_2} = e^{z_2 \ln z_1}} \quad (5)$$

For example $\boxed{i^i = e^{i \ln i} = e^{-\pi/2} = \text{REAL!}}$ (6)

CHANGE OF BASE:

e = base for natural logs. 10 = base for common logs ($1000 = 10^3$ etc.)

2 = base for binary numbers. For any base b

$$\boxed{a = b^{\log_b a}} \quad (7) \qquad 1000 = 10^{\log_{10}(1000)} = 10^3$$

$$3 = \log_{10}(1000)$$

A number in one base can be re-expressed in terms of another base:

$$y \equiv e^{\ln y} \equiv b^{\log_b y} \qquad b = \text{another base}$$

$$\hookrightarrow b = e^{\ln b}$$

$$\Rightarrow y = (e^{\ln b})^{\log_b y} = e^{\ln b \cdot \log_b y} \quad ; \ln \equiv \log_e \qquad (8)$$

But $y \equiv e^{\ln y} \;\hookrightarrow\; \ln y = \ln b \cdot \log_b y \Rightarrow \boxed{\log_b y = \frac{\ln y}{\ln b} = \frac{\log_e y}{\log_e b} \quad (9)}$

The two most common bases are e and 10

$\ln$ is the shorthand for $\log_e$ = "natural logs"

$\log_{10}$ = "common logs"

$\boxed{\text{Derivative of } (\text{Any Number})^{\text{any power}}:}$

$$\frac{d}{dx} x^p = \frac{d}{dx}(e^{\ln x})^p = \frac{d}{dx} e^{p \ln x} = e^{p \ln x} \cdot \underbrace{\frac{d}{dx}(p \ln x)}_{\frac{p}{x}} \qquad (10)$$

$$\Rightarrow \boxed{\frac{d}{dx} x^p = \frac{p}{x} \cdot \underbrace{e^{p \ln x}}_{x^p} = p\, x^{p-1}} \qquad (11)$$

This is the usual result, but now we have shown that it holds for any x and any p. Later we will show that it also holds when $x \to z$ = complex, and p is also complex.

This illustrates a general technique for generalizing simpler results.

TAYLOR SERIES

Suppose that we know the value of a function $f(x)$ at some point x, but we would like to be able to compute the value at a nearby point $(x+a)$. This can be done via a Taylor Series which can be written in a number of equivalent ways.

A formula (useful in Quantum Mechanics - see below!!) is

INCREMENT FORM

$$\boxed{f(x+a) = e^{a\frac{d}{dx}} f(x) = \left(1 + \frac{a}{1!}\frac{d}{dx} + \frac{a^2}{2!}\frac{d^2}{dx^2} + \dots\right) f(x)} \quad (1)$$

Equivalently by interchanging $x \leftrightarrow a$,

$$f(a+x) = e^{x\frac{d}{da}} f(a) = \left(1 + \frac{x}{1!}\frac{d}{da} + \frac{x^2}{2!}\frac{d^2}{da^2} + \dots\right) f(a) \quad (2)$$

If we now shift x by replacing $x \to x-a$ then $(2) \Rightarrow$

$$f(a+x-a) = f(x) = \left(1 + \frac{(x-a)}{1!}\frac{d}{da} + \frac{(x-a)^2}{2!}\frac{d^2}{da^2} + \dots\right) f(a) \quad (3)$$

TEXT (1.3, 4.4)

$$\boxed{\therefore\ f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots} \quad (4)$$

Both of these forms are useful in different contexts.

Connection to Quantum Mechanics: ($\hbar$ = Planck's constant)

In (1) write

$$a\frac{d}{dx} = \frac{ia}{\hbar} \cdot \underbrace{\frac{\hbar}{i}\frac{d}{dx}}_{\text{momentum operator} \equiv P} = \frac{iaP}{\hbar} \quad (5)$$

Hence (1) & $(5) \Rightarrow$

$$\boxed{e^{iaP/\hbar} f(x) = f(x+a)} \quad (6)$$

The momentum operator induces spatial translations

The fact that the momentum operator "moves" (translates) the function $f(x) \to f(x+a)$ is intimately associated with the

$\boxed{\text{HEISENBERG UNCERTAINTY PRINCIPLE}}$

$$\boxed{\Delta p \Delta x \gtrsim \hbar} \qquad (7)$$

The close connection between momentum conservation and spatial translations is part of NOETHER'S THEOREM.

APPLICATIONS OF TAYLOR SERIES:

[1]
$$f(x) = \frac{1}{1-x} \equiv \frac{1}{u} \qquad (8)$$

$$f(x) = f(a) + \frac{(x-a)}{1!} f^{(1)}(a) + \frac{(x-a)^2}{2!} f^{(2)}(a) + \ldots \qquad \text{choose } a=0 \qquad (9)$$

$$f^{(1)}(a) = f'(a) = \frac{df}{du} \cdot \frac{du}{dx}\bigg|_{x=0} = -\frac{1}{u^2} \cdot (-1)\bigg|_{x=0} = \frac{1}{(1-x)^2}\bigg|_{x=0} = +1 \qquad (10)$$

$$f^{(2)}(a) = f''(a) = \frac{-2}{(1-x)^3} \cdot (-1)\bigg|_{x=0} = +2 \qquad (11)$$

$$\therefore f(x) = \frac{1}{1-x} = \underbrace{f(0)}_{1} + \frac{(x-0)}{1!} \cdot 1 + \frac{(x-0)^2}{2!} \cdot 2 + \ldots \quad 1 + x + x^2 + \ldots \qquad (12)$$

We will show later that for $|x|<1$ this series converges. Hence for $|x|<1$ we can write

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n \qquad |x|<1 \qquad (13)$$

Also
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x|<1$$

[2] $\boxed{f(x) = \ln x}$

$$f(x) = f(a) + \frac{(x-a)}{1!} f^{(1)}(a) + \frac{(x-a)^2}{2!} f^{(2)}(a) + \cdots \qquad (14)$$

Again the Taylor series starts by knowing a value $\underline{a}$ at which we know the function. Here we can choose $x=1 \Rightarrow \ln 1 = 0$. Also from our previous results, for any value a

$$D^n \ln a = \frac{(-1)^{n+1}(n-1)!}{a^n} \xrightarrow{a=1} (-1)^{n+1}(n-1)! \nearrow 0! = 1 \qquad (15)$$

Hence the first few terms in the series expansion of $\ln x$ are

$$\ln x = \underbrace{\ln 1}_{0} + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \qquad (16)$$

This series converges for $|x-1| < 1 \Rightarrow 0 \leq x < 2$. Another useful form for the expansion of $\ln x$ follows from the replacement $y = x-1 \Rightarrow x = 1+y$. Then (16) $\Rightarrow$

$$\boxed{\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots} \qquad (17)$$

This formula is used in our derivation of The RULE OF 72: $\underline{\ln(1+y) \cong y}$

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[3] $\boxed{f(x) = (1+x)^n}$ Typical applications are for small x near 1 so we expand about $f(a)=1$, which corresponds to $\underline{x=a=0}$. Then

$$f(x) = \underbrace{f(a)}_{f(x=a)} + \frac{(x-a)}{1!} f'(x=a) + \frac{(x-a)^2}{2!} f''(x=a) + \cdots \qquad (18)$$

Note that (as a notation check) $f(x=a) = f(a) + 0 + 0 + 0 \cdots$ which is as expected.

$$f'(x) = \frac{df}{du}\cdot\frac{du}{dx} \quad ; u = 1+x \; ; \frac{du}{dx} = 1 \qquad (19)$$

$$\Rightarrow f'(x) = nu^{n-1}\cdot 1 = n(1+x)^{n-1} \Rightarrow f'(x=a=0) = n \qquad (20)$$

$$f''(x) = n(n-1)u^{n-2}\cdot 1 \Rightarrow f''(x=a=0) = n(n-1)\cdots \qquad (21)$$

Hence $$(1+x)^n = 1 + \frac{(x-0)}{1!}\cdot n + \frac{(x-0)^2}{2!}n(n-1) + \cdots \qquad (22)$$

$\therefore$ To leading order, $\boxed{(1+x)^n \cong 1+nx}$ (23)

More generally: $$(1\pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \qquad (24)$$

This series converges for $x^2 < 1$.

CHECK!

For $n=-1$, $$(1\pm x)^{-1} = \frac{1}{1\pm x} = 1 \mp x + x^2 \mp x^3 \qquad (25)$$

This agrees with the previous results in (13).

— — — — — —

The preceding results also hold if n is replaced by a rational power p, such as $p = 1/2$. This is useful, for example in the theory of relativity where we often encounter the expressions $\sqrt{1-v^2/c^2}$ and $1/\sqrt{1-v^2/c^2}$. Let $x = v^2/c^2$, then

$$\sqrt{1-v^2/c^2} \Rightarrow (1\mp x)^{1/2} = 1 \mp \tfrac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \mp \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 \cdots \qquad (26)$$

$$\therefore (1\mp x)^{1/2} = 1 \mp \tfrac{1}{2}x - \tfrac{1}{8}x^2 \mp \tfrac{1}{16}x^3 \cdots \qquad (27)$$

GR 1.112

Also: $$(1\mp x)^{-1/2} = \frac{1}{\sqrt{1\mp x}} = 1 \pm \tfrac{1}{2}x + \tfrac{3}{8}x^2 \pm \tfrac{5}{16}x^3 \cdots \qquad (28)$$

Hence $$\frac{mc^2}{\sqrt{1-v^2/c^2}} \cong mc^2\left(1+\tfrac{1}{2}\frac{v^2}{c^2}\right) \cong mc^2 + \tfrac{1}{2}mv^2 \qquad (29)$$

COMPARISON OF THE TWO FORMS OF THE TAYLOR SERIES

INCREMENT FORM $f(x) =$ known then

$$f(x+a) = e^{a\frac{d}{dx}} f(x) = \left(1 + \frac{a}{1!}\frac{d}{dx} + \frac{a^2}{2!}\frac{d^2}{dx^2} + \dots\right) f(x) \qquad (1)$$

$$f(x+a) = f(x) + \frac{a}{1!} f'(x) + \frac{a^2}{2!} f''(x) + \dots \qquad (2)$$

In this form we assume that we know the function $f(x)$ at some point x, and this allows us to evaluate the r.h.s. exactly. Then if we know $f(x)$ we can compute $f(x+a)$.

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any x — a specific value $(x=a)$

TAYLOR SERIES FORM

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \qquad (3)$$

In this form we again assume that we know the function $f(x)$, which now appears on the l.h.s. of (3). Here the main objective is to construct a series expansion of $f(x)$ in terms of the (presumably small) quantities $(x-a)$, $(x-a)^2$, …

A very useful compendium of formulas for series is:

I.S. Gradshteyn & I.M. Ryzhik

"TABLE OF INTEGRALS, SERIES AND PRODUCTS"

(Academic Press, 1980)