INTRODUCTION TO INTEGRAL CALCULUS

Loosely (speaking) integral calculus answers the question of what the inverse of differentiation is: For example

Differential Calculus: If $F(x) = x^3 + 17$ then $F'(x) = \frac{dF(x)}{dx} = 3x^2 \equiv f(x)$ (1)

Integral Calculus: Suppose I know the function $f(x)$. What function $F(x)$ is such that $F'(x) = dF(x)/dx = f(x)$. In this case we know that the answer is not unique, since any $F(x)$ of the form $F(x) = x^3 + C$ where C = constant will work. Here the answer is not yet unique since we have yet to specify the constant C. This can be done by adding more information in the form of initial conditions.

More on this later.

Why Do We Care???

In some sense the objective in combining mathematics and physics is to allow us to obtain new information from the information we already have. Sometimes this involves differentiation and sometimes this involves integration:

[A] Suppose we know the position of a car $x(t)$ on a highway as a function of time. We can then obtain new information by noting that $v(t) = dx(t)/dt \equiv$ Velocity and (2)

$a(t) = dv(t)/dt = d^2x(t)/dt^2 \equiv$ acceleration (3)

Suppose, however, that you know as an experimental fact that the acceleration of an object falling near the surface of the Earth is

$$\frac{d^2 x(t)}{dt^2} = g = 9.80\,m/s^2 = \text{constant} \qquad (4)$$

Your question is how far will an object fall in 1 second? To answer this question we wish to find $x(t)$, which when twice differentiated gives Eq. (4). By inspection we can see that

$$\boxed{x(t) = x_0 + v_0 t + \tfrac{1}{2} g t^2 \;;\; x_0, v_0 \text{ are constants}} \qquad (5)$$

Check: $\dfrac{dx(t)}{dt} = v_0 + gt \;;\; \dfrac{d^2 x(t)}{dt^2} = g$ (6)

As noted above, to fully determine $x(t)$ we need information on x_0 (the initial position at $t=0$) and v_0 (the initial velocity).

Although one can obtain (5) from (4) by inspection, integration is rarely this easy!! However, extensive tables are available for thousands of common functions $f(x)$ giving the functions $F(x)$ such that $dF(x)/dx \equiv F'(x) = f(x)$.

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Before considering the general case we start with the case $f(x) = f = \text{constant}$, as in the above gravity example. It is then obvious by inspection that the function $F(x)$ whose derivative $F'(x) = f(x) = f$ is given by

$$F(x) = F(x_0) + (x - x_0) f = \underbrace{[F(x_0) - x_0 f]}_{\text{constant}} + x \cdot f \qquad (7)$$

WHY DO WE CARE (continued)?

[1] Your Rental Car: Have you been speeding?

Many rental cars have accelerometers built in to "black boxes".

Using

Velocity

$$\frac{dv(t)}{dt} = a(t)$$

acceleration = known

This is the analog of $\frac{dF(x)}{dx} = f(x) =$ known

We can then use the knowledge of $a(t)$ to find

$$\boxed{v(t) - v(t=0) = \int_{t=0}^{t} a(t')dt'}$$

does this exceed posted speed limits?

[2] Inertial Guidance for Missles

Knowing $v(t)$ from the above, we can then calculate how far a missle has traveled. In analogy to the above

$$\boxed{x(t) - x(t=0) = \int_{t=0}^{t} v(t')dt'}$$

This allows the position of a missle, or commercial plane, to be determined internally by measuring $a(t)$, without being affected by clouds, etc.

Now the spirit of calculus is that over any short interval Δx, a smooth curve can be approximated by a straight line, so that the function $f(x)$ starting at $x = x_i$ can be approximated by $f(x_i)$ over the short interval Δx. Recall that if we have a function $f(x) = f =$ constant, the cumulative effect of this function varying from x_0 to x is given by

$$F(x) = F(x_0) + f\cdot(x - x_0) = F(x_0) + (x - x_0)f \qquad (8)$$

Evidently $f(x)$ governs the change in $F(x)$ even if $f(x)$ is varying, so over a small interval Δx we can write

$$\left.\frac{\Delta F(x)}{\Delta x}\right|_{x_i} \cong f(x_i) \Rightarrow \Delta F \cong f(x_i)\Delta x \qquad (9)$$

For example: If your Ferrari accelerates with $a(t)$, then the change in velocity Δv over some short time interval Δt is

$$\Delta v \cong a(t_i)\Delta t \qquad (10)$$

over that interval. The question we are then asking is what is the cumulative change in velocity over, say 10 seconds if $a(t)$ itself is changing?

Returning to (8) & (9) we want to find the cumulative change in $F(x)$ over the interval between x and x_0. So we break up the interval $(x - x_0)$ into small pieces $\Delta x = (x - x_0)/N$, so that over each piece $f(x) \cong f(x_i)$ is constant. Then

$$\boxed{F(x) \cong F(x_0) + \sum_{i=1}^{N} f(x_i)\Delta x} \qquad (11)$$

We next take the limit $N \to \infty$. In this limit $\Delta x \to dx$ and the approximation that $f(x) \cong f(x_i) \cong$ constant becomes exact.

We then define:

→ see discussion of "dummy variables"

$$\boxed{\lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x_i \equiv \int_{x_0}^{x} f(x')dx' \equiv \text{definite integral of } f(x) \text{ between } x \text{ and } x_0} \quad (12)$$

We note that the word "integral" conveys the idea of a cumulative effect. Referring back to Fig. 2.1 in the text, we see that each term in the sum represents the area of a rectangle of height $f(x_i)$ and width Δx_i so that in the end, the left hand side (l.h.s.) of (12) actually represents the cumulative area under the curve (i.e. between the curve and the x-axis).

Hence the definite integral has the general meaning of "the area under the curve" in one-dimension, or its generalization in higher dimensions (e.g. area inside a closed curve, or volume of a solid).

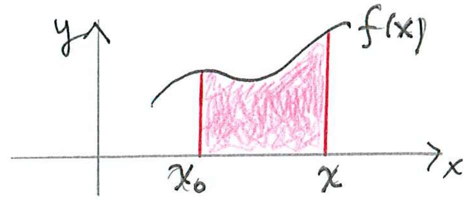
Combining (11) & (12) we can then write

$$\boxed{F(x) = F(x_0) + \int_{x_0}^{x} f(x')\,dx'} \quad (13)$$

HOW DO WE EVALUATE $\int_{x_0}^{x} f(x)\,dx$?

Side Comment: Using the same x symbol here is awkward, but we will return to this later when we introduce "dummy variables"

(A) Numerical Integration



For a function that is finite and well behaved we can evaluate the integral $\equiv$ area under the curve $f(x)$ between x_0 and x by literally breaking up the area into small rectangles, just as we did to define $\int_{x_0}^{x} f(x)\,dx$. There are many numerical algorithms which can do this to varying degrees of approximation. (An example is the Simpson algorithm).

(B) Analytic Evaluation

In many instances we can evaluate $\int_{x_0}^{x} f(x)\,dx$ exactly.

To see how return to (9):

$$\left.\frac{\Delta F}{\Delta x}\right|_{x_i} = f(x_i) \xrightarrow{\Delta x \to 0} \boxed{\frac{dF(x)}{dx} = f(x)} \text{ at any } x \qquad (14)$$

From (13)

$$F(x) = \underbrace{F(x_0)}_{\text{constant}} + \int_{x_0}^{x} f(x)\,dx \Rightarrow \qquad (15)$$

$$\boxed{\frac{dF(x)}{dx} = 0 + \frac{d}{dx}\int_{x_0}^{x} f(x')\,dx'} \qquad (16)$$

Comparing (14) and (16) we see that it must be the case that

$$\frac{d}{dx}\int_{x_0}^{x} f(x')\,dx' = f(x) \qquad (17)$$

This will become more meaningful if we introduce the "dummy variable" x' so that (17) reads

$$\boxed{\frac{d}{dx}\int_{x_0}^{x} f(x')\,dx' = f(x)} \qquad (18)$$

This is sometimes called the FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS. It expresses the fact that the change in the area under the curve, $dF(x)/dx$ at some point x is just $f(\ldots)$ evaluated at that point, namely $f(x)$.

Knowing this, we can return to (14) and write:

$$\frac{dF(x)}{dx} = f(x) = \text{known function} \qquad (19)$$

Then in many case we can GUESS what form $F(x)$ must have to make (19) true. For example, suppose we know that the functional form of $f(x)$ is $f(x) = x^2$. We might then guess that $F(x) = \frac{1}{3}x^3 + c \leftarrow$ constant; (20)

since $dF(x)/dx = \frac{1}{3}\cdot 3x^2 = x^2$ ✓

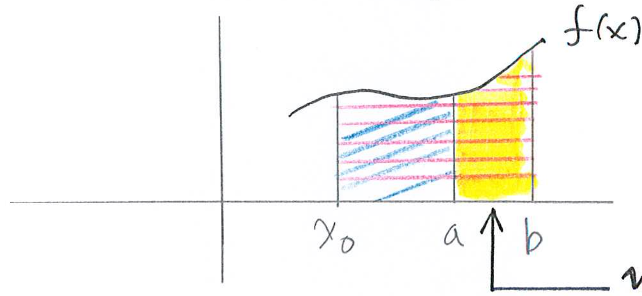
To finish the calculation of $F(x)$ we now want to evaluate the constant C.

From the discussion leading to (13) we see that $F(x)$ gives the "area under the curve $f(x)$" between x_0 and x. Clearly then when $x = x_0$ this area $= 0$, so we have to choose C to make this happen. Trivially we then find $C = -\frac{1}{3}x_0^3$.

Hence altogether the area under the curve $y = f(x) = x^2$ between x_0 and any variable point x is

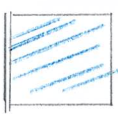
$$\boxed{F(x) = \frac{1}{3}x^3 - \frac{1}{3}x_0^3 \quad ; \quad F(x_0) = 0 \checkmark} \qquad (21)$$

In practice we are usually interested in the area under the curve between two points a and b which have some physical relevance for the problem at hand, in contrast to x_0 which is completely arbitrary:

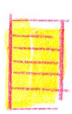


We are only interested in the highlighted region between a and b

From (13): $F(x) = F(x_0) + \int_{x_0}^{x} f(x)\,dx$

$\Rightarrow$ blue region $=$  $= F(a) = F(x_0) + \int_{x_0}^{a} f(x')\,dx'$ (22)

red region $=$  $= F(b) = F(x_0) + \int_{x_0}^{b} f(x')\,dx'$ (23)

$\Rightarrow$ highlighted region $=$  $= F(b) - F(a) = \int_{x_0}^{b} f(x')\,dx' - \int_{x_0}^{a} f(x')\,dx' = \int_{a}^{b} f(x')\,dx'$ (24)

The last step follows by looking at the figure and noting that we could have focused on the highlighted area immediately simply by choosing the arbitrary starting point x_0 to be $x_0 = a$. Then:

$$\text{highlighted area} \equiv F(b) - F(a) = \int_a^b f(x')dx' - \underbrace{\int_a^a f(x')dx'}_{=0} = \int_a^b f(x')dx' \qquad (25)$$

This gives the widely used result:

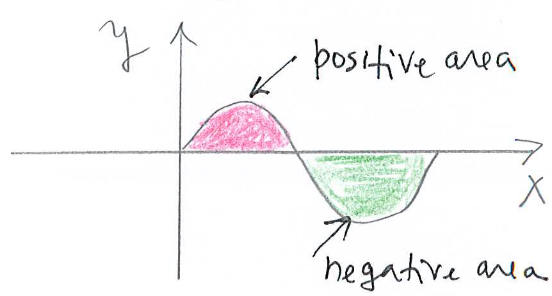
$$\boxed{\text{Area under curve between } a \text{ and } b \equiv F(b) - F(a) = \int_a^b f(x')dx'} \qquad (26)$$

Another useful result: From (24)

$$\int_{x_0}^b f(x')dx' - \int_{x_0}^a f(x')dx' = \int_a^b f(x')dx' \Rightarrow \qquad (27)$$

$$\boxed{\int_{x_0}^b f(x')dx' = \int_{x_0}^a f(x')dx' + \int_a^b f(x')dx'} \qquad (28)$$

<u>NOTE:</u> Although we think of physical areas as positive, the areas computed from (26) can be positive or negative (or zero!) since they are being referred to the x-axis



if $y(x) = \sin x$ then the total area for 1 cycle (as shown) $= 0$

Returning to (26) we write

$$F(b) - F(a) = \int_a^b f(x')\,dx' \qquad (29)$$

or

$$F(a) - F(b) = \int_b^a f(x')\,dx'$$

$$\left.\begin{array}{l} \text{But } \underbrace{F(a) - F(b)}_{\int_b^a f(x)\,dx} = -[\underbrace{F(b) - F(a)}_{\int_a^b f(x)\,dx}] \end{array}\right\} \Rightarrow \boxed{\int_b^a f(x')\,dx' = -\int_a^b f(x')\,dx'} \qquad (30)$$

Eq. (30) reinforces the fact that the areas defined by integrals can be positive or negative.

SUMMARY OF BASIC FORMULAS FOR INTEGRATION

$$F(x) = F(x_0) + \int_{x_0}^{x} f(x')\,dx' \qquad \text{Eq. (13)}$$

$$\frac{dF(x)}{dx} = f(x) = \frac{d}{dx}\int_{x_0}^{x} f(x')\,dx' \qquad \text{Eq. (18)}$$

[understand "dummy variables" like x']

$$\boxed{F(b) - F(a) = \int_{a}^{b} f(x')\,dx' \qquad \text{Eq. (26)}}$$

$$\int_{x_0}^{b} f(x')\,dx' = \int_{x_0}^{a} f(x')\,dx' + \int_{a}^{b} f(x')\,dx' \qquad (28)$$

→ In practice this is the most widely used formula.

HOW TO PROCEED:

Q1: Given a function $f(x)$, find the area between $f(x)$ and the x-axis (i.e. under the curve $f(x)$) bounded by $x=a$, $x=b$

A: [1] Begin by finding the function $F(x)$ having the property $dF(x)/dx = f(x)$, as in (18). $F(x)$ is called the "anti-derivative" or "indefinite integral". In so doing neglect any constant contribution to $F(x)$.

[2] Once $F(x)$ is known, simply evaluate $[F(x=b) - F(x=a)]$. This is your answer.

Q2: Given $f(x)$ how do we go "backwards" and find $F(x)$ so that $dF(x)/dx = f(x)$?

A: (a) In some cases $F(x)$ is obvious (see below)

(b) various tricks (see below)

(c) guess and check!

(d) consult tables.

EXAMPLES

[1] Find $[F(3)-F(1)]$ corresponding to the function $f(x)=x^2$:

$$F(3)-F(1) = \int_1^3 x^2\,dx \qquad (1)$$

(a) By inspection we note that if $F(x) = \frac{1}{3}x^3$ then $\frac{dF(x)}{dx} = \frac{1}{3}\cdot 3x^2 = x^2$

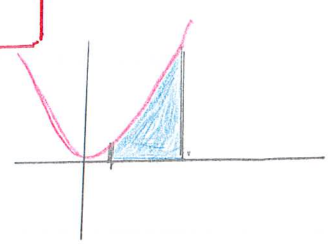
So $F(x) = \frac{1}{3}x^3$ works, and we cand drop any extra constant. (2)

(b) Then $F(3)-F(1) = \frac{1}{3}x^3\Big|_{x=3} - \frac{1}{3}x^3\Big|_{x=1} = \frac{1}{3}\cdot 3^3 - \frac{1}{3}1^3 = 9 - \frac{1}{3} = 8\frac{2}{3}$ (3)

Comment: "The trick" we used here is that for any polynomial or sum of polynomials we can write

$$\boxed{f(x) = x^n \Rightarrow F(x) = \frac{x^{n+1}}{n+1}} \qquad (4)$$

It often helps to sketch your solution



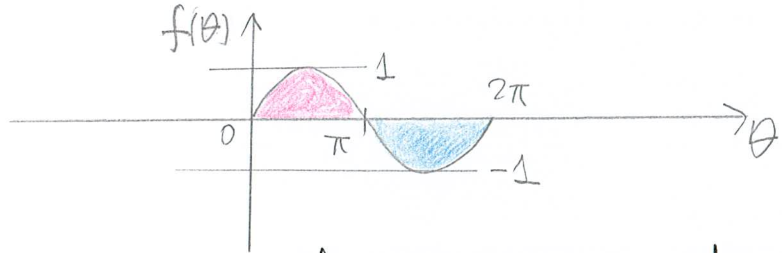
For a sum of polynomials treat each term separately using (4).

EXAMPLES (continued)

[2] Find $[F(\theta=2\pi) - F(\theta=0)]$ for $f(x) = \sin x$

$\Rightarrow f(\theta) = \sin\theta$ ↳ in radians

Here a picture really helps:



Here we recall that $\frac{d}{d\theta}\sin\theta = \cos\theta$; $\frac{d}{d\theta}\cos\theta = -\sin\theta$ (5)

Hence $f(\theta) = \sin\theta \Leftrightarrow F(\theta) = -\cos\theta \Rightarrow [F(\theta=2\pi) - F(\theta=0)]$ (6)

$-\cos(2\pi) - (-)\cos(0) = -1+1 = 0$

Hence the area "under" the curve $= 0$;

but this is because 1/2 the curve is above the horizontal axis, while the other half is below the horizontal axis.

[2'] Find $[F(\theta=2\pi) - F(\theta=\pi)] = -\cos(2\pi) - (-)\underbrace{\cos(\pi)}_{-1} = -2 < 0$ (7)

Note the fact that this area is negative agrees with the picture above.

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[3] Find $[F(2) - F(1)]$ for $f(x) = \ln x$

Here we note that $\frac{d}{dx}\ln x = \frac{1}{x} \Rightarrow \frac{d}{dx}\{x\ln x - x\} = 1\cdot\ln x + x\cdot\frac{1}{x} - 1 = \ln x$ (8)

Hence

$$f(x) = \ln x \Leftrightarrow F(x) = x\ln x - x$$

$\Rightarrow [F(2) - F(1)] = [x\ln x - x]_{x=2} - [x\ln x - x]_{x=1} = (2\ln 2 - 2) - (1\,\underbrace{\ln 1}_{0} - 1)$

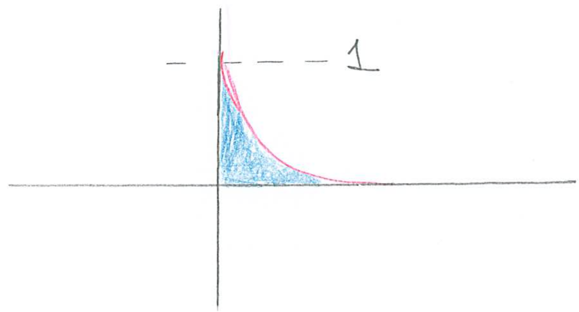
$= 2\ln 2 - 2 + 1$ (9)

$= 2\ln 2 - 1$

More Terminology: $F(x) = \int f(x)\,dx$ = "indefinite integral"

this gives the functional form of $F(x)$

[4] Find the area under the curve $f(x) = e^{-x}$ between the origin and ∞.



In our previous notation we want

$$F(\infty) - F(0) = \int_0^{\infty} f(x')\,dx' \quad ; \quad \underline{f(x) = e^{-x}} \qquad (10)$$

Step [1]: find $F(x)$ having the property $\frac{dF}{dx} = e^{-x}$. By inspection we note that

$$\frac{d}{dx} e^{-x} = -e^{-x} \Rightarrow \frac{d}{dx}(-e^{-x}) = e^{-x} \Rightarrow \underline{F(x) = -e^{-x}} \qquad (11)$$

Step [2]:
Hence the area under the curve $= F(\infty) - F(0) = \underbrace{-e^{-x}\Big|_{x\to\infty}}_{0} \underbrace{-(-)e^{-x}\Big|_{x=0}}_{+e^0 = 1}$

$$\Rightarrow \boxed{F(\infty) - F(0) = 1}$$

[4′] Repeat the previous problem with $\underline{f(x) = e^{-ax}}$ (12)

From the previous calculation we note that if $\underline{F(x) = -\frac{1}{a} e^{-ax}}$ (13)

then $\frac{dF(x)}{dx} = -\frac{1}{a} \frac{d}{dx}(e^{-ax}) = -\frac{1}{a}(-a)e^{-ax} = e^{-ax}$

$$\Rightarrow \boxed{F(\infty) - F(1) = -\frac{1}{a}\left[e^{-ax}\Big|_{x\to\infty} - e^{-ax}\Big|_{x=0} \right] = \frac{1}{a}} \qquad (14)$$

EXAMPLES (continued)

[4″] We repeat the previous calculation for $f(x) = e^{-ax}$ by the method of change of variables: We want to evaluate

$$F(\infty) - F(0) = \int_0^\infty f(x')dx' = \int_0^\infty e^{-ax'}dx'$$

Let $\underbrace{y = ax'}$ $\Rightarrow$ $dy = a\,dx'$ $\Rightarrow$ $dx' = \frac{1}{a}dy$;

Also: $x' = 0 \Rightarrow y = 0$; $x' = \infty \Rightarrow y = \infty$ $\Rightarrow \int_{x'=0}^{x'=\infty} e^{-ax'}dx' = \int_{y=0}^{y=\infty} e^{-y}\frac{1}{a}dy$

But this is just $\boxed{\frac{1}{a}\underbrace{\int_0^\infty e^{-y}dy}_{1} = \frac{1}{a}1}$ ⟵ Same as before!

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[5] Find the area under the curve between $[0, b]$ for the function $\boxed{f(x) = \frac{x}{\sqrt{x^2+a^2}}}$

We guess that the corresponding $F(x)$ is: $F(x) = \sqrt{x^2+a^2} \equiv u^{1/2}$

Check: $$\frac{dF(x)}{dx} = \frac{dF}{du}\cdot\frac{du}{dx} = \frac{1}{\cancel{2}}u^{-1/2}\cdot\cancel{2}x = \frac{x}{\sqrt{x^2+a^2}}$$

Hence what we want is $F(b) - F(0) \equiv \int_0^b \frac{x'dx'}{\sqrt{x'^2+a^2}} = \sqrt{x'^2+a^2}\Big|_{x=b} - \sqrt{x'^2+a^2}\Big|_{x=0}$

$$\Rightarrow \boxed{F(b) - F(0) = \sqrt{b^2+a^2} - a}$$

Some Useful Anti-Derivatives

$f(x)$	$\Leftrightarrow$ $F(x)$
x^n	$\dfrac{x^{n+1}}{n+1}$
e^x ; e^{ax}	e^x ; e^{ax}/a
$\ln x$	$x\ln x - x$
$1/x$	$\ln x$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{x}{\sqrt{x^2\pm a^2}}$	$\sqrt{x^2\pm a^2}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln(\cos x)$
$\cot x$	$\ln(\sin x)$
$\sin^2 x$	$\dfrac{1}{2}x - \dfrac{1}{4}\sin 2x$
$\cos^2 x$	$\dfrac{1}{2}x + \dfrac{1}{4}\sin 2x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln(\cosh x)$