INTEGRATION BY PARTS

This is another widely used technique which reduces the evaluation of certain anti-derivatives to ones we already know. We begin by defining

$$DF(x) \equiv \frac{dF(x)}{dx} = f(x) \qquad DG(x) = \frac{dG(x)}{dx} = g(x) \qquad (1)$$

We have previously shown that $D(FG) = DF \cdot G + F \cdot DG \quad (2)$
$= fG + Fg$

Suppose that for a given problem we are given ~~f(x)~~ and $g(x)$ and also $G(x)$; $F(x)$. Using (2) we can then write

$$\boxed{Fg = D(FG) - Gf} \qquad (3)$$

It follows from (3) that
$$\int_{x_1}^{x_2} F(x')g(x')\,dx' = \int_{x_1}^{x_2} \frac{d}{dx'}\left(F(x')G(x')\right)dx' - \int_{x_1}^{x_2} G(x')f(x')\,dx' \qquad (4)$$

Consider the term shown by ~~~

$$= \boxed{\int_{x_1}^{x_2} \frac{d}{\cancel{dx'}}(FG)\,\cancel{dx'} \equiv FG\Big|_{x_1}^{x_2}} \qquad (5)$$

This simply states that the process of differentiation and the process of integration offset each other

Combining (4) and (5) we arrive at

$$\boxed{\int_{x_1}^{x_2} F(x')g(x')\,dx' = \underbrace{F(x')G(x')\Big|_{x_1}^{x_2}}_{\text{"SURFACE TERM"}} - \int_{x_1}^{x_2} G(x')f(x')\,dx'} \qquad (6)$$

PARTIAL INTEGRATION FORMULA

As we will see in an example below, this is particularly useful in situations where the "surface term" vanishes.

Comments:

(1) Note that on the r.h.s. (right hand side) of (6) we have to know $G(x)$ obtained from $g(x)$ so that $dG(x)/dx = g(x)$. We have also noted that $G(x)$ is undetermined up to a constant. This means that for any $G(x)$ we can find a new $G(x)$ by adding a constant c. Here we show explicitly that this constant does not change the answer in (6). If we replace $G(x) \longrightarrow G(x)+c$ in (6) then

$$\int_{x_1}^{x_2} F(x')g(x')dx' = \left[F(x')G(x') + F(x')\cdot c\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} [G(x')+c]f(x')dx' \qquad (7)$$

The extra terms due to c are:

$$\text{extra terms} = c\,F(x')\Big|_{x_1}^{x_2} - c\underbrace{\int_{x_1}^{x_2} f(x')dx'}_{F(x')\big|_{x_1}^{x_2}} = 0 \Rightarrow \underline{\text{no change due to } c} \qquad (8)$$

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(2) NOTATION: In many texts you will see the partial integration formula written as

$$\int (u)(dv) = uv - \int (v)(du)$$

or simply:

$$\boxed{\int u\,dv = uv - \int v\,du} \qquad (9)$$

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(3) WHY IS PARTIAL INTEGRATION USEFUL?

Example [1] $\displaystyle\int_{x_1}^{x_2} x'\cos x'\,dx'$ $\qquad \underbrace{\text{let } u = x'}_{du = 1dx'} \quad \underbrace{dv = \cos x'\,dx'}_{V = +\sin x'}$ (10)

$dv = +\cos x'dx'$ ✓

$$\Rightarrow \boxed{\int_{x_1}^{x_2} x'\cos x'dx' = x'\sin x'\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \sin x'\cdot 1dx'} \qquad (11)$$

The first term in (11) – the "surface term" can be evaluated trivially for any x_2, x_1.

The remaining integral is:

$$-\int_{x_1}^{x_2} \sin x'\,dx' = \cos x'\Big|_{x_1}^{x_2} \qquad (12)$$

Hence altogether we find: $$\boxed{\int_{x_1}^{x_2} \underbrace{x'\cos x'}_{f(x')}\,dx' = \Big[\underbrace{x'\sin x' + \cos x'}_{F(x')}\Big]_{x_1}^{x_2}} \qquad (13)$$

Check: we can (and must!) verify that $dF(x')/dx' = f(x')$

$$\frac{d}{dx'}\left[x'\sin x' + \cos x'\right] = 1\cdot\cancel{\sin x'} + x'\cos x' - \cancel{\sin x'} = x'\cos x' \checkmark$$

Example [2] – Problem 2.1.3 of text

Consider the function $$F(n) = \int_0^\infty \underbrace{x'^n}_{u}\, \underbrace{e^{-x'}dx'}_{dv} = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du \qquad (14)$$

If $u = x'^n \Rightarrow du = n(x')^{n-1}dx'$

If $dv = e^{-x'}dx' \Rightarrow v = -e^{-x'}$

Then (14) $\Rightarrow$ $$F(n) = \underbrace{x'^n(-e^{-x'})\Big|_0^\infty}_{\equiv 0} - \int_0^\infty (-)e^{-x'}\, n(x')^{n-1}dx' \qquad (15)$$

→ This is one of the powerful uses of partial integration: it happens sometimes that the surface term $\equiv 0$.

From (15) we then have: $$F(n) = (-)(-)n \underbrace{\int_0^\infty e^{-x'}(x')^{n-1}}_{\equiv F(n-1)} \qquad (16)$$

$$\Rightarrow \quad \boxed{F(n) = n\,F(n-1)} \qquad (17)$$

Continuing: $F(n-1) = (n-1)F(n-2) \Rightarrow F(n) = n(n-1)F(n-2)$ (18)

Continuing in this manner we find in the end

$$\boxed{F(n) = n(n-1)(n-2)(n-3)\ldots 3\times 2\times 1 = n!} \quad (19)$$

Later you will learn of the gamma function $\Gamma(n)$:

$$\boxed{\Gamma(n) = F(n-1) = (n-1)!} \quad (20)$$

From the preceding we then see that $\Gamma(n)$ can be expressed as

Integral Representation of $\Gamma(n)$

$$\boxed{\Gamma(n) = (n-1)! = \int_0^{\infty} (x')^{n-1} e^{-x'} dx'} \quad (21)$$

One advantage of the Integral Representation is that it allows us to define $\Gamma(n)$ for any value of n: positive, negative, zero, complex.

For example:

$$(n-1)! \xrightarrow{n=1} 0! = \int_0^{\infty} (x')^{1-1} e^{-x'} dx' = \int_0^{\infty} e^{-x'} dx' = -e^{-x'}\Big|_0^{\infty} = -e^{-\infty} + e^{0} = 1 \quad (22)$$

Hence $\boxed{0! = 1}$ (23)

We can use this result to evaluate $\Gamma(n)$ for negative n:

Note that:

$$n! = n(n-1)! \Rightarrow (n-1)! = \frac{n!}{n} \quad (23)$$

$$n=1 \Rightarrow (n-1)! = \frac{n!}{n} = \frac{1!}{1} = 1$$

$$n=0 \Rightarrow (-1)! = \frac{0!}{0} = \frac{1}{0} = \infty \quad (24)$$

$$n=-1 \Rightarrow (-2)! = \frac{-1!}{-1} = \frac{+\infty}{-1} = -\infty$$

etc. $\Rightarrow \Gamma(n) \rightarrow \infty$ when $n \leq -1$

Evaluation of Gaussian Integrals

This is a class of integrals of great importance which recur often in physics. Additionally their evaluation exhibits added tricks that we can use for other integrals.

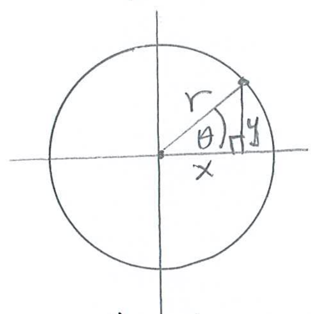
Consider $I \equiv \int_{-\infty}^{\infty} e^{-x^2}\,dx$ (1) (here we are dropping the primes on x, since there is no chance of confusion)

The first trick is to write I^2 as

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \quad (2)$$

Here x and y are dummy variables: The key point is that we want to make clear that each integral is to be treated separately, at least initially.

Next we transform to polar coordinates:



$x = r\cos\theta \qquad y = r\sin\theta$

$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

Returning to (2): $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta\, e^{-r^2}$ (3)

we will show this later!

from $\int d\theta$

From (3): $\boxed{I^2 = 2\pi \cdot \int_0^{\infty} r e^{-r^2}\,dr}$ (4)

Note that by allowing θ to range from 0 to 2π, while r ranges from 0 to ∞, the range of integration extends over all space, just as the integration over $dx\,dy$ does in (2).

In Eq.(4) make the substitution $s = r^2$

Then $\frac{ds}{dr} = 2r \Rightarrow ds = 2r\,dr$ or $r\,dr = \frac{ds}{2}$ (5)

Now Eq.(4) becomes: $I^2 = 2\pi \int_0^\infty e^{-s} \frac{ds}{2}$ (6)

Note that since $s = r^2$, $r = 0 \Rightarrow s = 0$, and $r \to \infty \Rightarrow s \to \infty$, so that the integration limits in (6) remain $[0, \infty]$ just as in the original expression in (4). Hence

$I^2 = \pi \underbrace{\int_0^\infty e^{-s} ds}_{1^*}$ ← this is the integral we have already evaluated on p.43*

Thus $I^2 = \pi \cdot 1 \Rightarrow$ $\boxed{I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$ (7)

Moreover, since e^{-x^2} has the same numerical value for $\pm x$, it follows that negative values contribute exactly ½ of the result in (7), so that

$\boxed{\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}}$ (8)

The function e^{-x^2} is called a GAUSSIAN (or "NORMAL") function and describes the usual "bell-shaped" curve. For many purposes we "normalize" the curve, so that the area under it is 1. So in the end

$$\boxed{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1}$$

GAUSSIAN INTEGRALS (continued)

In practice we would like to have available normalized Gaussians which are more-or-less-sharply peaked. Consider the function e^{-y^2/a^2} and the integral

$$J = \int_{-\infty}^{\infty} e^{-y^2/a^2}\,dy \quad ; \quad \text{let } y^2/a^2 \equiv x^2 \Rightarrow y^2 = a^2x^2 \Rightarrow 2y\,dy = a^2 2x\,dx \qquad (1)$$

To understand the last step start with $y^2 = a^2x^2$ and let $x \to x+\Delta x$. This produces a change Δy in y given by $(y+\Delta y)^2 = a^2(x+\Delta x)^2$

$$\Rightarrow y^2 + 2y\Delta y + \cancel{(\Delta y)^2} = a^2(x^2 + 2x\Delta x + \cancel{(\Delta x)^2}) \qquad \times \Rightarrow \text{higher order}$$

$$\therefore \cancel{y^2} + 2y\Delta y = \cancel{a^2x^2} + a^2 2x\Delta x \quad ; \quad \text{but } y^2 = a^2x^2 \qquad (2)$$

$$\Rightarrow 2y\Delta y \cong a^2 2x\Delta x \Rightarrow 2y\,dy = a^2 2x\,dx$$

$$\Rightarrow dy = \frac{x}{y}a^2dx = a\,dx$$

Hence in (1): $$J = \int_{-\infty}^{\infty} e^{-x^2}a\,dx = a\int_{-\infty}^{\infty} e^{-x^2}dx = a\sqrt{\pi} \qquad (3)$$

Finally $$\boxed{\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2/a^2}dy = 1} \qquad (4)$$

In this form we can vary a so as to achieve Gaussian functions which are broader or narrower, while remaining normalized.

From (4) we can define $b = 1/a^2 \Rightarrow f(b) = \int_{-\infty}^{\infty} dy\, e^{-by^2} = \sqrt{\pi/b}$ (5)

$$\frac{df(b)}{db} = -\int_{-\infty}^{\infty} e^{-by^2}\cdot y^2 = \frac{d}{db}\sqrt{\frac{\pi}{b}} = -\frac{1}{2}\sqrt{\pi/b^3}$$

Hence finally: $$\boxed{\int_{-\infty}^{\infty} e^{-by^2}\cdot y^2\,dy = \frac{1}{2}\sqrt{\frac{\pi}{b^3}}} \qquad (6)$$

MORE INTEGRATION TRICKS

The previous discussion of Gaussian integrals introduced several techniques for integration: substitution (change of variables), use of dummy variables, and differentiation with respect to a parameter. We now exhibit other examples:

[1] Text p.44 $\quad F(x_1,x_2) = \int_{x_1}^{x_2} \frac{x'^2\,dx'}{(x'^3+4)^2}$ $\quad$ (1)

In this case we can begin by dropping the primes so that $x' \to x$. We can do this here because x does not appear anywhere else in the integral.

The trick here is to try the substitution $x^3 = u \Rightarrow \frac{du}{dx} = 3x^2$ $\quad$ (2)

In the spirit of $dy/dx = 1/dx/dy$ etc. we can substitute for dx in (1):

$$\frac{du}{dx} = 3x^2 \Rightarrow \quad dx = \frac{du}{3x^2} \quad \text{or} \quad dx = \frac{dx}{du}\,du = \frac{1}{3x^2}\,du \qquad (3)$$

$$\Rightarrow F(x_1,x_2) = \int_{x_1}^{x_2} \frac{x^2 \cdot \left(\frac{1}{3x^2}\right) du}{(u+4)^2} = \int_{x_1}^{x_2} \frac{du}{3(u+4)^2} \qquad (4)$$

Comments: up to this point the "trick" was to recognize that the various powers of x in the numerator & denominator of (1) would combine to give a simple expression in terms of u.

Note also that we should be re-expressing the upper and lower limits x_2, x_1 in terms of u_2 & u_1, but we can do this later.

To continue we carry out another substitution $v = u+4 \Rightarrow dv = du$

$$\Rightarrow J = \int_{x_1}^{x_2} \frac{dv}{3v^2} = -\frac{1}{3}\frac{1}{v}\Bigg|_{x_1}^{x_2} \qquad (5)$$

Finally we straighten out the limits: Since (5) is now expressed in terms of v we have two choices:
We can re-express the limits in terms of v, or we can leave the limits as is and re-express v in terms of x. Using the latter approach

$$v = u + 4 \; ; \; u = x^3 \Rightarrow v = x^3 + 4. \text{ Hence altogether}$$

$$F(x_1, x_2) = \int_{x_1}^{x_2} \frac{x^2 dx}{(x^3+4)^2} = -\frac{1}{3} \left. \frac{1}{(x^3+4)} \right|_{x_1}^{x_2} = \ldots \text{ etc.} \qquad (6)$$

Importance of the JACOBIAN:

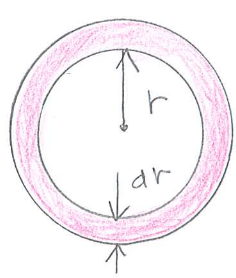
One of the key steps in the preceding discussion is the substitution

$$x^3 = u \Rightarrow \Rightarrow \boxed{dx = \frac{dx}{du} \cdot du} \qquad \frac{dx}{du} \equiv J\left(\frac{x}{u}\right) = \text{JACOBIAN} \quad (7)$$

Whenever a change of variables is made, the Jacobian will enter to give a proper weighting of the infinitesimal volume element. For example the 3-dimensional volume element in a spherically symmetric world is

$$dx\,dy\,dz \longrightarrow 4\pi r^2 dr$$

; the factor of r^2 expresses the fact that for a given thin shell dr, there is more volume the bigger r is.



In the general case: $$\int_{x_1}^{x_2} f(x)\,dx = \int_{u(x_1)}^{u(x_2)} f(x(u))\, J\left(\frac{x}{u}\right) du$$

Example [2] Text 2.2.1

$$F(x_1,x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{a^2-x^2}}\,dx \qquad 0 < x < \pi/2 \qquad (1)$$

Here we follow the text and substitute $x = a\sin\theta$: (2)

$$\frac{dx}{d\theta} = a\cos\theta \Rightarrow dx = a\cos\theta\, d\theta$$

$$\Rightarrow F(x_1,x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta\, d\theta = \int_{x_1}^{x_2} \frac{1}{\cancel{a}\underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta}} \cdot \cancel{a}\cos\theta\, d\theta \qquad (3)$$

$$\therefore F(x_1,x_2) = \int_{x_1}^{x_2} d\theta = \theta \Big|_{x_1 = \cdots}^{x_2 = \cdots} \qquad (4)$$

But we can write $\theta = \sin^{-1}(x/a)$ from (2)

$$\text{Hence: } \boxed{F(x_1,x_2) = \sin^{-1}(x_2/a) - \sin^{-1}(x_1/a)} \qquad (5)$$

Example [3] $F(x_1,x_2) = \int_0^\infty \frac{1}{x^2+a^2}\,dx$; let $x = a\tan\theta$

$$x^2+a^2 = a^2(\tan^2\theta + 1) = \frac{a^2}{\cos^2\theta} \qquad (6)$$

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = a\frac{1}{\cos^2\theta} \Rightarrow F(x_1,x_2) = \int \frac{\frac{a}{\cos^2\theta}\,d\theta}{\frac{a^2}{\cos^2\theta}} = \frac{1}{a}\int d\theta \qquad (7)$$

$$\Rightarrow dx = \frac{a}{\cos^2\theta}\,d\theta$$

limits: $x = a\tan\theta \Rightarrow$ $x = 0 \Leftrightarrow \theta = 0$, $x = \infty \Leftrightarrow \theta = \pi/2$

$$\Rightarrow \boxed{F(x_1,x_2) = \frac{1}{a}\int_0^{\pi/2} d\theta = \pi/2a} \qquad (8)$$

We will later show how to obtain the same result by contour integration in the complex plane.

Example [4]: Text 2.2.7

Evaluate $I = \int_0^1 2^x \, dx$; using $2 = e^{\ln 2}$

we avail ourselves of the knowledge of how to integrate e^x.

Note that $I = \int_0^1 (e^{\ln 2})^x dx = \int_0^1 e^{x \ln 2} dx$; let $y = x \ln 2$

$\Rightarrow \frac{dy}{dx} = \ln 2 \Rightarrow dx = \frac{1}{\ln 2} dy$

$$\Rightarrow I = \int e^y \cdot \frac{dy}{\ln 2} = \frac{1}{\ln 2} e^y \Big|_{x=0}^{x=1} = \frac{1}{\ln 2}\left[e^{x \ln 2}\right]_0^1 = \frac{1}{\ln 2}\left[e^{1 \ln 2} - 1\right]$$

$$\Rightarrow \boxed{I = \frac{1}{\ln 2}[2-1] = \frac{1}{\ln 2}} \qquad (9)$$

Example [5]: Text 2.2.27

Consider $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$; We have already shown that $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ (10)

Starting with (10): $\frac{d}{da}\int_0^\infty \frac{1}{x^2+a^2} dx$; let $u = x^2 + a^2$; (11)

$$I = \frac{d}{da}\int_0^\infty + \frac{1}{u(a)} dx = \int_0^\infty \frac{-1}{u^2}\left(\frac{du}{da}\right) \cdot dx = -\int_0^\infty \frac{1}{(x^2+a^2)^2} \cdot 2a\,dx \qquad (12)$$

$$= -2a \int_0^\infty \frac{dx}{(x^2+a^2)^2}$$

On the other hand, we have from (10):

$\frac{d}{da}\left(\frac{\pi}{2a}\right) = -\frac{\pi}{2a^2}$ $\Rightarrow$ Combining this with (12) $\Rightarrow$ $\frac{-\pi}{2a^2} = -2a \int_0^\infty \frac{dx}{(x^2+a^2)^2}$ (13)

Hence finally: $\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}}$ (14)

CALCULUS OF MANY VARIABLES

Since we live in a world of 3+ spatial dimensions (+ = string theory!) we want to generalize the preceding results beyond 1-dimension. Following the text we consider, initially, a function of 2 variables x, y such as the temperature $T(x,y)$ at a point (x,y) in this room. We then introduce the concept of the PARTIAL DERIVATIVE: Simply stated, this is the derivative with respect to one of the variables, while the other is held constant:

$$\frac{\partial f}{\partial x} \equiv \frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \qquad (1)$$

EXAMPLE: The volume of a cylinder is $V = \pi R^2 H$, where R is the radius and H is its height. We can ask how the volume changes when we separately change the radius and height:

$$\frac{\partial V(R,H)}{\partial R} = 2\pi R H \quad ; \quad \frac{\partial V(R,H)}{\partial H} = \pi R^2 \qquad (2)$$

A manufacturer wishing to increase the volume of a cylindrical can could then do this by changing R or H, and (2) tells her how to do it. Thus if the plan is to change R then

higher order terms

$$\boxed{V(R_0+\Delta R, H_0) = V(R_0, H_0) + \left.\frac{\partial V}{\partial R}\right|_{R_0, H} \times \Delta R + \dots} \qquad (3)$$

Suppose now that the manufacturer wishes to have the option of changing both R and H. Intuitively we guess that this is just the sum of the separate effects of changing $R \to R+\Delta R$ and $H \to H+\Delta H$. So we guess that

$$\boxed{V(R_0+\Delta R, H_0+\Delta H) - V(R_0,H_0) \cong \left.\frac{\partial V}{\partial R}\right|_{R_0,H_0} \otimes \Delta R + \left.\frac{\partial V}{\partial H}\right|_{R_0,H_0} \otimes \Delta H} \tag{4}$$

This turns out to be correct, if we neglect all the higher-order terms. We can prove this more generally, following the text:

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \cong \underbrace{f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0)}_{\partial f/\partial y|_{x_0+\Delta x, y_0} \otimes \Delta y} + \underbrace{f(x_0+\Delta x, y_0) - f(x_0,y_0)}_{\partial f/\partial x|_{x_0,y_0} \otimes \Delta x} \tag{5}$$

$$\Rightarrow \quad \cong \partial f/\partial y|_{x_0,y_0} \otimes \Delta y + \partial f/\partial x|_{x_0,y_0} \otimes \Delta x \tag{6}$$

Notice that in the last we are justified in dropping the Δx in $x_0+\Delta x, y_0$ (so that we are evaluating at x_0, y_0), because retaining this extra Δx would only produce higher order terms such as $\Delta x \Delta y$.

HIGHER ORDER DERIVATIVES

Taking a partial derivative produces a new function, which itself can be further differentiated, as in the above example:

Following the text consider $f = x^3 + 2xy^2$

$$\frac{\partial f}{\partial x} \equiv f_x = 3x^2 + 2y^2 \quad ; \quad \frac{\partial f}{\partial y} \equiv f_y = 4xy$$ FIRST DERIVATIVES

$$\frac{\partial^2 f}{\partial x^2} \equiv f_{xx} = 6x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad ; \quad \frac{\partial^2 f}{\partial y^2} \equiv f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 4x \qquad (7)$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 4y \quad ; \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4y \qquad (8)$$

Note in (8) that $f_{xy} = f_{yx}$; this is a general result which holds for all continuous functions.

As in the case of a single variable, these derivatives determine where a function has maxima and minima.

Problem 3.1.2: Evaluate the first and second derivatives for

$$f(x,y) = x^3 + x^2y^5 + y^4$$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^5 \quad ; \quad \frac{\partial f}{\partial y} = 5x^2y^4 + 4y^3$$

$$f_{xx} = 6x + 2y^5 \quad ; \quad f_{yy} = 20x^2y^3 + 12y^2$$

$$f_{xy} = 10xy^4 \qquad f_{yx} = 10xy^4$$