CONTRAVARIANT = COVARIANT FOR CARTESIAN VECTORS

- $x_i' = a_{ij} x_j \Rightarrow a_{ij} = \dfrac{\partial x_i'}{\partial x_j}$ old notation (1)

$x^{i'} = a^i_j x^j \Rightarrow a^i_j = \dfrac{\partial x^{i'}}{\partial x^j}$ new notation (2)

Invert (2): $a^m_i x^{i'} = \underbrace{a^m_i a^i_j}_{\delta^m_j} x^j = x^m$ $\boxed{\begin{aligned} &\Rightarrow x^m = a^m_i x^{i'} \\ &\Rightarrow a^m_i = \frac{\partial x^m}{\partial x^{i'}} \end{aligned}}$ (3)

In (3) replace m by j $\Rightarrow$ $\boxed{a^j_i = \dfrac{\partial x^j}{\partial x^{i'}}}$ (4)

- Compare this to (2): $a^i_j = \dfrac{\partial x^{i'}}{\partial x^j} \overset{i \Leftrightarrow j}{\Longrightarrow} \boxed{a^j_i = \dfrac{\partial x^{j'}}{\partial x^i}}$ (5)

Finally, comparing (4) & (5) we see that

$$\boxed{\frac{\partial x^j}{\partial x^{i'}} = \frac{\partial x^{j'}}{\partial x^i}} \quad (6)$$

From Eq. (6) we see that it makes no difference whether the primed coordinates are in the numerator or denominator, which is to say that CONTRAVARIANT = COVARIANT

MANIPULATING TENSORS

a) Contraction ("generalized trace")

$$T'^{\mu\rho} = T'^{\mu\rho\nu}_{\ \ \ \ \nu} \quad \text{(sum over repeated index } \nu) \tag{1}$$

Show that this is a 2ND rank tensor

$$T'^{\mu\nu\rho}_{\ \ \ \ \lambda}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\gamma}} \frac{\partial x^{\epsilon}}{\partial x'^{\lambda}} T^{\alpha\beta\gamma}_{\ \ \ \ \epsilon}(x) \tag{2}$$

Set $\lambda = \nu$ and sum:

$$T'^{\mu\rho\nu}_{\ \ \ \ \nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} \underbrace{\frac{\partial x'^{\nu}}{\partial x^{\gamma}} \frac{\partial x^{\epsilon}}{\partial x'^{\nu}}} T^{\alpha\beta\gamma}_{\ \ \ \ \epsilon}(x) \tag{3}$$

$$\frac{\partial x^{\epsilon}}{\partial x^{\gamma}} \equiv \delta^{\epsilon}_{\gamma} \quad \text{(key step)} \tag{4}$$

$$\text{Hence: } \boxed{T'^{\mu\rho\nu}_{\ \ \ \ \nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} T^{\alpha\beta\gamma}_{\ \ \ \ \gamma}} \tag{5}$$

This shows that $T'^{\mu\rho\nu}_{\ \ \ \ \nu}(x')$ transforms as a 2ND rank contravariant tensor.

b) Raising & Lowering Indices

- Define $S^{\mu\rho}_{\nu\lambda\sigma} \equiv g_{\nu\lambda} T^{\mu\rho}_{\sigma}$ = 5TH RANK MIXED TENSOR
- Then set $\mu = \nu$ & sum $\Rightarrow S^{\mu\rho}_{\mu\lambda\sigma} = \boxed{S^{\rho}_{\lambda\sigma} = g_{\mu\lambda} T^{\mu\rho}_{\sigma}}$

Hence effect of $g_{\mu\lambda}$ is to lower an index of the original tensor $T^{\mu\rho}_{\sigma}$. This is important when keeping track of indices (an example will follow later — generalized Laplacian)

- Indices can also be raised using the inverse tensor to $g_{\mu\nu} \equiv g^{\lambda\mu}$:

$$g^{\lambda\mu}(x)\, g_{\mu\nu}(x) = \delta^{\lambda}_{\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq 0 \end{cases}$$

TENSOR DENSITIES

- We have previously defined $g(x) = \det g_{\mu\nu}(x)$ (1)

Since $g_{\mu\nu}(x)$ is a 2ND rank covariant tensor it transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x) \qquad (2)$$

We can regard $\partial x^\rho/\partial x'^\mu \equiv f^\rho_\mu$ as the matrix relating x' and x in which case Eq. (2) can be written in the form

$$g'_{\mu\nu} = f^\rho_\mu\, g_{\rho\sigma}\, f^\sigma_\nu \qquad (3)$$

- Taking the determinant of both sides of (3) we get

$$\underbrace{\det g'_{\mu\nu}}_{g'} = \det\left(f^\rho_\mu\, g_{\rho\sigma}\, f^\sigma_\nu\right) = (\det f^\rho_\mu)\underbrace{(\det g_{\rho\sigma})}_{g}(\det f^\sigma_\nu) \qquad (4)$$

where we have used $\det(ABC) = (\det A)(\det B)(\det C)$ (5)

Hence from (4):

$$\boxed{g'(x') = (\det f)^2 g(x) \equiv \left|\frac{\partial x}{\partial x'}\right|^2 g(x)} \qquad (6)$$

JACOBIAN OF TRANSFORMATION FROM $x \to x'$

- Since $g(x)$ is a scalar [it has no indices] the "expected" transformation law would be

$$g'(x') = g(x) \qquad (7)$$

without the additional factor $(\det f)^2 = \left|\frac{\partial x}{\partial x'}\right|^2$.

The presence of the additional factor $\left|\frac{\partial x}{\partial x'}\right|^2$ makes $g(x)$ not just a scalar but a Scalar density.

More generally a quantity which transforms as a tensor except for additional factors of $|\partial x'/\partial x|^W$ is a tensor density of weight W.

Note that: $$\left|\frac{\partial x}{\partial x'}\right| = \left|\frac{\partial x'}{\partial x}\right|^{-1} \qquad (8)$$

Combining (8) & (6) we have

$$\boxed{g'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-2} g(x)} \qquad (9)$$

Hence $g(x)$ is a scalar density of weight $W=-2$.

We can show that any tensor of weight W can be expressed as a product of an ordinary tensor multiplied by a factor $g^{-W/2}$.

To see this let $\mathcal{T}^{\mu}_{\nu}(x)$ be a tensor density of weight W so that

$$\mathcal{T}'^{\mu}_{\nu}(x') = \left|\frac{\partial x'}{\partial x}\right|^{W} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \mathcal{T}^{\rho}_{\sigma}(x) \qquad (10)$$

$$\therefore [g'^{W/2}]\mathcal{T}'^{\mu}_{\nu}(x') = \underbrace{\left[\left|\frac{\partial x'}{\partial x}\right|^{-2} g(x)\right]^{W/2}\left|\frac{\partial x'}{\partial x}\right|^{W}}_{g(x)^{W/2}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \mathcal{T}^{\rho}_{\sigma}(x) \qquad (11)$$

$$\therefore \left\{g'^{W/2}_{(x')} \mathcal{T}'^{\mu}_{\nu}(x')\right\} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \left\{g^{W/2}(x) \mathcal{T}^{\rho}_{\sigma}(x)\right\} \Rightarrow \qquad (12)$$

$$g^{W/2}_{(x)} \mathcal{T}^{\rho}_{\sigma}(x) \sim T^{\rho}_{\sigma}(x) \rightarrow \text{an ordinary tensor}$$

ANOTHER SPECIAL TENSOR

Levi-Civita tensor $\epsilon^{\mu\nu\lambda\kappa}$

$$\epsilon^{\mu\nu\lambda\kappa} = \begin{cases} +1 & \text{even permutation of } 1234 \; (x^4 = ict) \\ -1 & \text{odd} \quad " \quad " \quad " \\ 0 & 2 \text{ indices equal} \end{cases}$$

This symbol plays an important role in formulating Maxwell's equations in 4-dimensional notation. It allows $\vec{E}$ & $\vec{B}$ to be expressed in terms of a single tensor $F_{\mu\nu}$.

<u>Notation:</u> $F_{\mu\nu} = F_{\nu\mu}$ (Symmetric tensor)

$F_{\mu\nu} = -F_{\nu\mu}$ (antisymmetric tensor)

Let $A_{\mu\nu}$ be any tensor. Then

$$\boxed{A_{\mu\nu} = \underbrace{\frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})}_{\text{SYMMETRIC}} + \underbrace{\frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})}_{\text{ANTISYMMETRIC}}}$$

Transformation of the Levi-Civita Symbol $\epsilon^{\mu\nu\lambda\kappa}$:

Recall: $\epsilon^{1234} = +1$ = even permutations of 1234

$\epsilon^{2134} = -1$ = even permutations of 2134

= odd permutations of 1234

$\epsilon^{\mu\nu\lambda\kappa} = 0$ when 2 indices are equal

To show that $\epsilon^{\mu\nu\lambda\kappa}$ is a tensor density consider the expression

$$\Xi^{\rho\sigma\eta\xi} \equiv \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\lambda}} \frac{\partial x'^{\xi}}{\partial x^{\kappa}} \epsilon^{\mu\nu\lambda\kappa} \qquad (1)$$

We can see that $\Xi^{\rho\sigma\eta\xi}$ must be proportional to $\epsilon^{\rho\sigma\eta\xi}$ by noting that (for example) $\Xi^{\rho\rho\eta\xi} = 0$. This follows immediately by noting from (1) that setting $\rho = \sigma$ on the r.h.s. of (1) has the effect of making the coefficient of $\epsilon^{\mu\nu\lambda\kappa}$ <u>SYMMETRIC</u> in $\mu \leftrightarrow \nu$, whereas $\epsilon^{\mu\nu\lambda\kappa}$ itself is <u>ANTISYMMETRIC</u>, and hence their product vanishes. We can thus write

$$\Xi^{\rho\sigma\eta\xi} = \underset{\uparrow \text{constant}}{C} \cdot \epsilon'^{\rho\sigma\eta\xi} \qquad (2)$$

To evaluate C write:

$$\Xi^{1234} \equiv C\,\epsilon'^{1234} = \frac{\partial x'^{1}}{\partial x^{1}} \frac{\partial x'^{2}}{\partial x^{2}} \frac{\partial x'^{3}}{\partial x^{3}} \frac{\partial x'^{4}}{\partial x^{4}} \overset{+1}{\epsilon^{1234}} \qquad (3)$$

$$+ \underbrace{\frac{\partial x'^{1}}{\partial x^{2}} \frac{\partial x'^{2}}{\partial x^{1}} \frac{\partial x'^{3}}{\partial x^{3}} \frac{\partial x'^{4}}{\partial x^{4}}}_{\sigma} \overset{-1}{\epsilon^{2134}} + \ldots \text{ all remaining permutations of } 1234$$

It can be seen that the expression on the r.h.s. of (3) is just the determinant of the transformation from $x \to x'$!

$$\text{r.h.s.} = \begin{vmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^4}{\partial x^1} \\ \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^4}{\partial x^2} \\ \cdots & \cdots & \frac{\partial x'^3}{\partial x^3} & \frac{\partial x'^4}{\partial x^3} \\ \cdots & \cdots & \cdots & \frac{\partial x'^4}{\partial x^4} \end{vmatrix} \qquad (4)$$

From (3) & (4) we thus see that

$$\mathfrak{X}^{1234} \equiv C \underbrace{\epsilon'^{1234}}_{+1} = \left|\frac{\partial x'}{\partial x}\right| \cdot 1 \Rightarrow \boxed{C = \left|\frac{\partial x'}{\partial x}\right| \quad (5)}$$

Combining the previous results we find

$$\mathfrak{X}^{\rho\sigma\eta\xi} = \left|\frac{\partial x'}{\partial x}\right| \epsilon'^{\rho\sigma\eta\xi} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\lambda}\frac{\partial x'^\xi}{\partial x^\kappa}\, \epsilon^{\mu\nu\lambda\kappa} \qquad (6)$$

Hence

$$\boxed{\epsilon'^{\rho\sigma\eta\xi} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\lambda}\frac{\partial x'^\xi}{\partial x^\kappa}\, \epsilon^{\mu\nu\lambda\kappa} \quad (7)}$$

This establishes that $\epsilon^{\mu\nu\lambda\kappa}$ is a 4TH rank tensor density of weight $W=-1$

THE AFFINE CONNECTION [See WEINBERG "Gravitation & Cosmology"]

- The affine connection $\Gamma^{\lambda}_{\mu\nu}(x)$ is NOT a tensor. In fact its importance derives precisely from the fact that this is so.
- The affine connection is part of the formalism for COVARIANT DIFFERENTIATION:

$$\underbrace{\frac{\partial}{\partial x^{\lambda}} V^{\mu}(x)}_{\text{not a tensor}} \longrightarrow \underbrace{\underbrace{\frac{\partial}{\partial x^{\lambda}}}_{\text{not a tensor}} V^{\mu}(x) + \underbrace{\Gamma^{\mu}_{\lambda\nu}}_{\text{not a tensor}} V^{\nu}(x)}_{\text{a tensor}} = D_{\lambda} V^{\mu}(x) \quad (1)$$

We show that the "bad" parts of $\frac{\partial}{\partial x^{\lambda}} V^{\mu}$ which make it not a tensor, exactly cancel against the "bad" parts of $\Gamma^{\mu}_{\lambda\nu} V^{\nu}$ so that their sum is indeed a tensor.

MOTIVATING THE AFFINE CONNECTION:

From elementary physics we know that (Newton 2ND Law)

$$\underset{\substack{\updownarrow \\ \text{"inertial"} \\ \text{coordinate system}}}{\text{no forces}} \Rightarrow \underbrace{\frac{d^2\vec{X}}{dt^2}}_{\text{3-dim}} \equiv 0 \longrightarrow \frac{d^2\xi^{\alpha}}{d\tau^2} \qquad d\tau \leftrightarrow dt \quad (2)$$

$$c^2 d\tau^2 = c^2 dt^2 - (d\vec{x})^2 = -\eta_{\alpha\beta}\, d\xi^{\alpha}\, d\xi^{\beta}$$

QUESTION: What does "Newton II" look like in a non-inertial coordinate system?

ANSWER: New "forces" arise (centrifugal, coriolis ...)

The affine connection describes these new "forces"

Define

$$\xi^\alpha = \xi^\alpha(x^\mu) \quad \text{or} \quad x^\mu = x^\mu(\xi^\alpha) \tag{3}$$

ξ^α ↑ inertial, x^μ ↑ non-inertial

$$(2) \Rightarrow \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau}\right) = 0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} \cdot \frac{dx^\mu}{d\tau}\right) \tag{4}$$

↑ introduces new cood system

$$= \underbrace{\left(\frac{d}{d\tau}\frac{\partial \xi^\alpha}{\partial x^\mu}\right)}\left(\frac{dx^\mu}{d\tau}\right) + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} \tag{5}$$

→ what we want: acceleration in new cood system

$$\frac{\partial}{\partial x^\mu}\frac{d\xi^\alpha}{d\tau} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \xi^\alpha}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\nu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\nu}\underbrace{\frac{\partial}{\partial x^\mu}\left(\frac{dx^\nu}{d\tau}\right)}_{\frac{d}{d\tau}\Big(\underbrace{\frac{\partial x^\nu}{\partial x^\mu}}_{\delta^\nu_\mu}\Big) = 0} \tag{6}$$

Altogether: (4)–(6) ⇒

$$0 = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} \tag{7}$$

→ What we want

Multiply each term in this equation by $\partial x^\lambda/\partial \xi^\alpha$ using

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu \Rightarrow \tag{8}$$

$$\boxed{\begin{gathered} 0 = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \delta^\lambda_\mu\frac{d^2x^\mu}{d\tau^2} \\ 0 = \frac{d^2x^\lambda}{d\tau^2} + \underbrace{\left(\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\right)}_{\equiv \Gamma^\lambda_{\mu\nu} = \text{AFFINE CONNECTION}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \end{gathered}} \tag{9}$$

Eq. (9) can be written in the form:

$$\boxed{\frac{d^2 x^\lambda}{d\tau^2} = - \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \qquad (10)$$

Compare this to the usual formulation of Newton II

$$\frac{d^2 \vec{x}}{dt^2} = \frac{\vec{F}}{m} \quad \leftrightarrow \quad \boxed{\frac{d^2 x^i}{dt^2} = \frac{F^i}{m}} \qquad (11)$$

We see that $\Gamma^\lambda_{\mu\nu}$ plays the role of the external forces acting on a system: Later we will see that since $\Gamma^\lambda_{\mu\nu}$ enters into the formula for the covariant derivative, <u>We can interpret covariant differentiation as introducing forces in a natural way</u> into a free particle Hamiltonian or Lagrangian. This is one reason why $\Gamma^\lambda_{\mu\nu}$ is important.

<u>Relation Between $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$:</u>

Start with an earlier equation: $\begin{pmatrix}1 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & 1\end{pmatrix}$ $x^4 = ict$

$$g_{\mu\nu}(x) = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}(\xi) \qquad (12)$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \cdot \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \eta_{\alpha\beta} \qquad (13)$$

$\to \Gamma^\rho_{\lambda\mu} \frac{\partial \xi^\alpha}{\partial x^\rho}$ $\qquad\qquad$ $\to \Gamma^\rho_{\lambda\nu} \frac{\partial \xi^\beta}{\partial x^\rho}$

Hence
$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu}\underbrace{\left(\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}\right)}_{\to\, g_{\rho\nu}} \qquad (14)$$

$$+\ \Gamma^\rho_{\lambda\nu}\underbrace{\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\eta_{\alpha\beta}\right)}_{g_{\mu\rho}} \qquad (15)$$

$$\therefore \boxed{\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu}\, g_{\rho\nu} + \Gamma^\rho_{\lambda\nu}\, g_{\mu\rho}} \qquad (16)$$

For H.W.: You will show that proceeding similarly,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2\, g_{\rho\nu}\,\Gamma^\rho_{\lambda\mu} \qquad (17)$$

Hence finally, using $g^{\nu\sigma} g_{\rho\nu} = \delta^\sigma_\rho$ (18)

"METRIC COMPATIBILITY CONDITION"

$$\boxed{\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right)} \qquad (9)$$

not a tensor — a tensor — not tensors

<u>NOTES:</u> (a) $\Gamma^\sigma_{\lambda\mu} = \Gamma^\sigma_{\mu\lambda}$ [See Eq. (9) above] (10)

(b) In an inertial coord system

$$g_{\mu\nu} \to \eta_{\mu\nu} = \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \qquad (11)$$

$$\Rightarrow \Gamma^\sigma_{\lambda\mu} \equiv 0$$