Comments About $R^{\mu}_{\lambda\beta\alpha}$:

1) $R^{\mu}_{\lambda\beta\alpha} \neq 0 \Rightarrow$ Space has intrinsic curvature

2) It is convenient to express $R^{\cdots}_{\cdots}$ in terms of all covariant (lower) indices by lowering the index λ. Then:

$$\boxed{\begin{aligned} R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\nu\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\kappa\lambda}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) \\ + g_{\eta\sigma}\left(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right) \end{aligned}}$$

3) In terms of $R_{\lambda\mu\nu\kappa}$ the following relations hold:

a) $R_{(\lambda\mu)(\nu\kappa)} = R_{(\nu\kappa)(\lambda\mu)}$ (symmetry)

b) $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu}$ (antisymmetry)

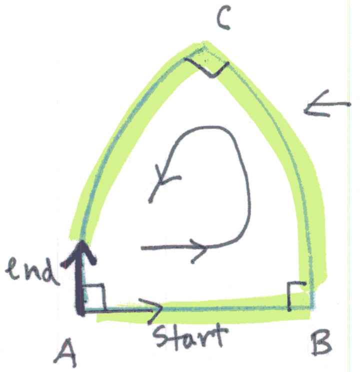
c) $R_{\lambda(\mu\nu\kappa)} + R_{\lambda(\kappa\mu\nu)} + R_{\lambda(\nu\kappa\mu)} = 0$ (cyclicity)

Physical Picture of Non-Commutativity

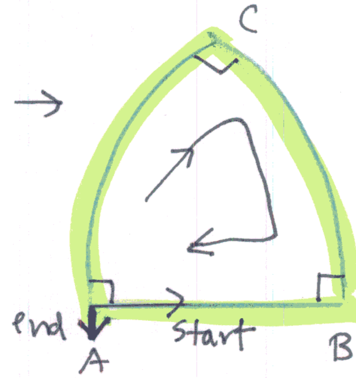
Recall Taylor series formula:

$$e^{a\frac{\partial}{\partial x}}\psi(x) = \psi(x+a) \quad \Big\} \text{ derivatives} \Rightarrow \text{translations}$$

Consider translations on a sphere:



← octant of sphere →



D_α = translation from $A \to C$ along curve ABC

D_β = translation from $A \to C$ along the curve AC

The fact that $[D_\alpha, D_\beta] \neq 0$ then reflects the fact that translations along an intrinsically curved surface do not commute.

COVARIANT EXPRESSIONS FOR GRAD, CURL, DIVERGENCE

CURL: Recall that $V_{\mu;\nu} = \dfrac{\partial V_\mu}{\partial x^\nu} - \Gamma^{\rho}_{\mu\nu} V_\rho \equiv \partial_\nu V_\mu - \Gamma^{\rho}_{\mu\nu} V_\rho$ (1)

COVARIANT CURL $\equiv V_{\mu;\nu} - V_{\nu;\mu} = \left(\partial_\nu V_\mu - \cancel{\Gamma^{\rho}_{\mu\nu} V_\rho}\right) - \left(\partial_\mu V_\nu - \cancel{\Gamma^{\rho}_{\nu\mu} V_\rho}\right)$ (2)

$= \partial_\nu V_\mu - \partial_\mu V_\nu \equiv V_{\mu,\nu} - V_{\nu,\mu}$ (3)

Hence the covariant curl is the same as the usual expression.

DIVERGENCE

Start with $V^{\mu}_{;\nu} = V^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\rho} V^\rho \Rightarrow V^{\mu}_{;\mu} = \underbrace{V^{\mu}_{,\mu}}_{\text{usual expression}} + \Gamma^{\mu}_{\mu\rho} V^\rho$ (4)

We will simplify this expression which eventually leads to the covariant (or generalized) LAPLACIAN: Recall that

$\nabla^2 \phi = \vec{\nabla}\cdot(\vec{\nabla}\phi) \rightarrow$ covariant divergence

Return to (4): (See p.55(9)) $\Gamma^{\sigma}_{\lambda\mu} = \dfrac{1}{2} g^{\nu\sigma}\left\{\dfrac{\partial g_{\mu\nu}}{\partial x^\lambda} + \dfrac{\partial g_{\lambda\nu}}{\partial x^\mu} - \dfrac{\partial g_{\mu\lambda}}{\partial x^\nu}\right\} = \Gamma^{\sigma}_{\mu\lambda}$ (5)

Hence $\Gamma^{\mu}_{\mu\rho} = \dfrac{1}{2} g^{\nu\mu}\left\{\dfrac{\partial g_{\mu\nu}}{\partial x^\rho} + \dfrac{\partial g_{\rho\nu}}{\partial x^\mu} - \dfrac{\partial g_{\mu\rho}}{\partial x^\nu}\right\}$

$$= \frac{1}{2} g^{\nu\mu}\frac{\partial g_{\mu\nu}}{\partial x^\rho} + \underbrace{\underbrace{\frac{1}{2} g^{\nu\mu}}_{\substack{\text{symm}\\ \text{in } \nu\leftrightarrow\mu}} \underbrace{\left[\frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\rho\mu}}{\partial x^\nu}\right]}_{\substack{\text{anti-symm}\\ \text{in } \nu\leftrightarrow\mu}}}_{0} = \frac{1}{2} g^{\nu\mu}\frac{\partial g_{\mu\nu}}{\partial x^\rho}$$

Let us focus on $\Gamma^{\mu}_{\mu\rho} = \frac{1}{2} g^{\nu\mu}_{(x)} \frac{\partial g_{\mu\nu}(x)}{\partial x^{\rho}}$ (1)

Since $g^{\nu\mu}_{(x)} g_{\mu\lambda}(x) = \delta^{\nu}_{\lambda}$, $g^{\nu\mu}$ is the matrix inverse of $g_{\mu\lambda}$

To simplify $\Gamma^{\mu}_{\mu\rho}$ we prove the following identity for a matrix M:

$$\boxed{\mathrm{Tr}\left\{ M^{-1}(x) \frac{\partial}{\partial x^{\rho}} M(x) \right\} = \frac{\partial}{\partial x^{\rho}} \ln \det M(x)} \quad (2)$$

First prove the following identity for a matrix A:

$$\boxed{\det e^{A} = e^{\mathrm{Tr} A}} \quad (3)$$

We prove this for the case that A can be diagonalized by a matrix U:

$$U^{-1} A U = B = \text{diagonal} \quad (4)$$

Hence: $\mathrm{Tr}\, B = \mathrm{Tr}\,(U^{-1} A U) = \mathrm{Tr}\,(U U^{-1} A) = \mathrm{Tr} A$ (5)

Also: $\det B = \det(U^{-1} A U) = \det U^{-1} \cdot \det A \cdot \det U = \det(U^{-1} U) \cdot \det A$
$= \det A$ (6)

Consider next $\det e^{B} = \det\left\{ \mathbb{1} + B + \frac{1}{2!} B^2 + \dots \right\}$ (7)

$$= \det\left\{ U^{-1} U + U^{-1} A U + \frac{1}{2!} U^{-1} A U U^{-1} A U + \dots \right\} \quad (8)$$

$$= \det\left\{ U^{-1} \left[\mathbb{1} + A + \frac{1}{2!} A^2 + \dots \right] U \right\} = \det\left\{ U^{-1} e^{A} U \right\} = \det e^{A} \quad (9)$$

$$\therefore \boxed{\det e^{B} = \det e^{A}} \quad (10)$$

Since B is diagonal we have:

$$\det e^A = \det e^B = \det\left\{\begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & \ddots\end{pmatrix} + \begin{pmatrix}b_{11} & & \\ & b_{22} & \\ & & b_{33} \ddots\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}b_{11}^2 & & \\ & b_{22}^2 & \\ & & \ddots\end{pmatrix}\cdots\right\}$$

$$= \det\left\{\begin{pmatrix}(1+b_{11}+\frac{1}{2!}b_{11}^2+\dots) & 0 & 0 & \\ 0 & (1+b_{22}+\frac{1}{2!}b_{22}^2+\dots) & 0 & \cdots\\ 0 & 0 & 1+b_{33}+\frac{1}{2!}b_{33}^2 & \\ \vdots & \vdots & \vdots & \end{pmatrix}\right\} \quad (11)$$

$$= \det\left\{\begin{pmatrix}e^{b_{11}} & 0 & 0 & \cdots\\ 0 & e^{b_{22}} & 0 & \\ 0 & 0 & e^{b_{33}} & \\ \vdots & \vdots & \vdots & \end{pmatrix}\right\} = e^{b_{11}} e^{b_{22}} e^{b_{33}} \cdots$$

$$= e^{b_{11}+b_{22}+b_{33}+\dots}$$

$$= e^{\mathrm{Tr}\,B} \quad (12)$$

Hence

$$\left.\begin{matrix}\det e^B & = & e^{\mathrm{Tr}\,B}\\ \parallel & & \parallel\\ \det e^A & & e^{\mathrm{Tr}A}\end{matrix}\right\} \Rightarrow \boxed{\det e^A = e^{\mathrm{Tr}A}} \quad (13)$$

To apply this to $\Gamma^{\mu}_{\mu\rho}$ let $\boxed{B = \ln M}$ (14)

(12), (13) & (14) $\Rightarrow$

$$\left.\underbrace{\det e^{\ln M}}_{\det M} = e^{\mathrm{Tr}\ln M}\right\} \boxed{\det M = e^{\mathrm{Tr}\ln M}} \quad (15)$$

This leads to another useful identity: Take ln of both sides:

$$\det M = e^{\mathrm{Tr}\ln M} \Rightarrow \boxed{\ln \det M = \mathrm{Tr}\ln M} \quad (16)$$

This identity applies even when $M = M(x)$. So, differentiate with respect to x:

$$\frac{\partial}{\partial x^\rho} \ln \det M(x) = \frac{\partial}{\partial x^\rho} \mathrm{Tr} \ln M = \mathrm{Tr} \frac{\partial}{\partial x^\rho} \ln M$$
$$= \mathrm{Tr}\left\{ M^{-1}(x) \frac{\partial}{\partial x^\rho} M(x) \right\} \tag{17}$$

Recall that for an ordinary function $f(x)$

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \frac{df(x)}{dx} \;;\; \text{for a matrix } \frac{1}{f} \to M^{-1} \tag{18}$$

Returning to p.70(1):

$$\Gamma^{\mu}_{\mu\rho} = \frac{1}{2} \mathrm{Tr}\left\{ (g_{\mu\nu})^{-1} \frac{\partial}{\partial x^\rho} \overbrace{g_{\mu\nu}}^{=M} \right\} = \frac{1}{2} \frac{\partial}{\partial x^\rho} \ln \underbrace{\det(g_{\mu\nu})}_{\equiv g(x)} \tag{19}$$

$$\boxed{\Gamma^{\mu}_{\mu\rho} = \frac{\partial}{\partial x^\rho} \frac{1}{2} \ln g(x) = \frac{\partial}{\partial x^\rho} \ln \sqrt{g(x)} = \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^\rho} \sqrt{g(x)}} \tag{20}$$

$$g(x) = \det g_{\mu\nu}(x)$$

Return to the covariant divergence:

$$V^{\mu}_{;\mu}(x) = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^{\mu}_{\mu\rho} V^\rho = \frac{\partial V^\mu}{\partial x^\mu} + \left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} \right) V^\mu \tag{21}$$

$$\boxed{\therefore\; V^{\mu}_{;\mu}(x) = \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g(x)}\, V^\mu(x) \right)} \tag{22}$$

NOTE! This expression is covariant even though it is expressed in terms of a conventional partial derivative $\partial/\partial x^\mu$.

Application: Laplacian in 3-dimensional Spherical coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \quad (1)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \; ; \; g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

$$ds^2 = (dr, d\theta, d\phi)\left(\downarrow \right)\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = g_{\mu\nu} dx^\mu dx^\nu \quad (2)$$

$$g = \det g_{\mu\nu} = r^4 \sin^2\theta \; ; \; \sqrt{g} = r^2 \sin\theta \quad (3)$$

<u>Laplacian</u> $\nabla^2 \Phi = \vec{\nabla} \cdot (\vec{\nabla}\Phi) \equiv D_\lambda (D^\lambda \Phi) = \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g} D^\lambda \Phi)$ (4)

<u>Key Step</u>: $D^\lambda \Phi = g^{\lambda\nu} D_\nu \Phi = g^{\lambda\nu} \partial_\nu \Phi$ (5)

conventional partial derivatives are covariant vectors

Also: $D_\nu \Phi = \partial_\nu \Phi$ since $\Gamma^\lambda_{\mu\nu}$ has no way to enter

Hence $D^\lambda \Phi$ has the following components:

$$D^\lambda \Phi = \left(1 \cdot \partial_r \Phi, \; \frac{1}{r^2} \partial_\theta \Phi, \; \frac{1}{r^2 \sin^2\theta} \partial_\phi \Phi \right) \quad (6)$$

From (4): $\nabla^2 \Phi = D_\lambda (D^\lambda \Phi) = \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g} D^\lambda \Phi)$ (7)

$$= \frac{1}{r^2 \sin\theta} \left\{ \partial_r (r^2 \sin\theta \cdot \partial_r \Phi) + \partial_\theta (\cancel{r^2} \sin\theta \cdot \frac{1}{\cancel{r^2}} \partial_\theta \Phi) + \partial_\phi (\cancel{r^2 \sin\theta} \cdot \frac{1}{\cancel{r^2 \sin^2\theta}} \partial_\phi \Phi) \right.$$

Hence finally: $$\boxed{\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2}} \quad (8)$$

LINEAR ALGEBRA

TERMINOLOGY:

Ⓐ Group: A system $(G, \cdot)$ with elements $a, b, \ldots \in G$ and a closed operation $\cdot$ such that

1) $a \cdot b = c \in G \quad \forall a, b$
2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ — associativity*
3) $\exists\, i : \; i \cdot a = a \cdot i = a \;\; \forall a$ — identity element
4) For $\forall a \; \exists\, a^{-1} \ni \; a \cdot a^{-1} = a^{-1} \cdot a = i$ — inverse element

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5) If $a \cdot b = b \cdot a$ the group is commutative (Abelian)

Examples

a) Real numbers (excluding 0) $\quad \cdot = \times \quad i = 1$, $x^{-1} = 1/x$

b) Integers with $\cdot = +$ (Abelian)

c) Rotations of a sphere (non-Abelian)

* d) An example of a non-associative operation is

$$\left.\begin{array}{l} a \cdot b = a \times b + a + b \\ \text{(check for yourself!)} \end{array}\right\} \begin{array}{l} (a \cdot b) \cdot c - a \cdot (b \cdot c) \\ = 2 \times (c - a) \end{array}$$

(B) Field : A system $\{F, +, \cdot\}$ satisfying the following axioms:

a) $\{F, +\}$ is an Abelian Group ; identity $= 0$

b) $\{F_0, \cdot\}$ is an Abelian Group ; identity $= i$

↳ all x except $x = 0$

c) For $a, b, c \in F$ $\quad a\cdot(b+c) = a\cdot b + a\cdot c$

distributivity

Examples:

rational numbers, real numbers, complex numbers with $+ =$ addition and $\cdot =$ multiplication

There are other structures we can define such as RINGS, but for our purposes GROUPS & FIELDS suffice.