POWER SERIES IN A VARIABLE x

The more common case is when we confront a series which depends on some variable x as well as on n. For example, we began this discussion with the series

$e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$. In the case of power series (a series in powers of x), the condition for convergence may depend on both x and n.

Consider the generic series $S(x) = \sum_{n=0}^{\infty} a_n x^n$ (1)

Using the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \underbrace{\left| \frac{a_{n+1}}{a_n} \right|}_{r} \qquad (2)$$

Hence

$$R(x) \equiv \lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \cdot r \qquad (3)$$

It follows that the condition for convergence $|R(x)| < 1 \Rightarrow |x| r < 1$

or $\boxed{|x| < \frac{1}{r}}$ (4)

NOTE: MY NOTATION FOR R(X) IS DIFFERENT FROM THAT OF THE TEXT

If the condition for convergence is $|x| < |x_{max}|$, then $|x_{max}|$ is called the radius of convergence of the series in which case the series only converges for x in the range

$$\boxed{-x_{max} < x < x_{max}} \qquad (5)$$

If we now apply the ratio test to the series for e^x we find

$$R(x) \equiv \frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x \cdot n!}{(n+1)!} = \frac{x}{n+1} \qquad (6)$$

Hence for any finite value of the variable x we see that $\lim_{n\to\infty} R(x) = 0$. Moreover, since $\sin x$, $\cos x$, $\sinh x$, and $\cosh x$ are defined in terms of e^x, these series converge for all values of x.

EXAMPLE 2: On the other hand consider the series for $\frac{1}{1-x}$: (7)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \Rightarrow \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} = x \qquad (8)$$

$\Rightarrow R(x) = x \Rightarrow$ The series for $\frac{1}{1-x}$ converges for $|x| < 1$ only.

EXAMPLE 3:

As an extreme case, consider $\sum_{n=1}^{\infty} n!\, x^n \Rightarrow a_n = n!\, x^n \Rightarrow$ (9)

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)x \;; \text{ as } n \to \infty,\ \frac{a_{n+1}}{a_n} \longrightarrow \infty \qquad (10)$$

Hence this series has a radius of convergence of zero. This means that there is no value (or range of values) of x for which this series converges.

HOWEVER!! This does not mean that such a series is useless! For some purposes a series which does not formally converge is in fact useful for numerical purposes, as in the case of an ASYMPTOTIC SERIES (ASYMPTOTIC EXPANSION).

APPLICATIONS OF CONVERGENCE TESTS

Text problem 4.3.1 : Find the radius of convergence for each series

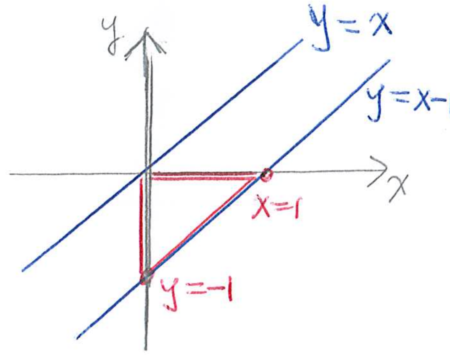
(i) $\sum_1^{\infty} \frac{x^{2n}}{2^n n^2} \Rightarrow R_n \equiv \frac{x^{2(n+1)}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{x^{2n}} = \frac{x^2}{2}\left(\frac{n}{n+1}\right)^2$

$R \equiv \lim_{n\to\infty} R_n = \frac{x^2}{2}$; $|R|<1 \Rightarrow \frac{x^2}{2}<1 \Rightarrow \boxed{x<\sqrt{2}}$

(ii) $\sum_1^{\infty} \frac{(x-1)^n}{n} \Rightarrow R_n \equiv \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} = (x-1)\frac{n}{n+1}$

$R = \lim_{n\to\infty} R_n = (x-1) \Rightarrow |x-1|<1$

From this figure $|x-1|<1 \Rightarrow \boxed{|x|<1}$



(iii) $\sum_1^{\infty} \frac{x^{3n}}{n} \Rightarrow R_n \equiv \frac{x^{3(n+1)}}{n+1} \cdot \frac{n}{x^{3n}} = x^3 \cdot \frac{n}{n+1}$

$R = \lim_{n\to\infty} R_n = x^3 \Rightarrow |x^3|<1 \Rightarrow \boxed{|x|<1}$

(iv) $\sum_1^\infty \frac{x^n}{n(n+1)} \Rightarrow R_n = \frac{x^{n+1}}{\cancel{(n+1)}(n+2)} \cdot \frac{n\cancel{(n+1)}}{x^n} = x\,\frac{n}{n+2}$ (1)

$R = \lim_{n\to\infty} R_n = x \Rightarrow \boxed{|x|<1}$ (2)

(v) $\sum_1^\infty \frac{n^{1/n}}{x^n}$; here the text suggests that we use $y = 1/x$ as the variable, initially

$\Rightarrow \sum \Rightarrow \{y^n n^{1/n}\}$

$R_n(y) \Rightarrow n \to n+1$ etc.

$$\Rightarrow R_n(y) = \frac{y^{n+1}(n+1)^{\frac{1}{n+1}}}{y^n\, n^{\frac{1}{n}}} = y\,\frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} \quad (3)$$

Here we can simplify the result by noting that as $n \to \infty$ we can approximate

$$(n+1)^{\frac{1}{n+1}} \cong n^{\frac{1}{n+1}} \quad (4)$$

$$\Rightarrow R(y) = \lim_{n\to\infty} R_n(y) = y\,\frac{n^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} = y\,n^{\left(\frac{1}{n+1}-\frac{1}{n}\right)} \quad (5)$$

$$\frac{1}{n+1} - \frac{1}{n} = \frac{\cancel{n}-\cancel{n}-1}{n(n+1)} = \frac{-1}{n(n+1)} \cong \frac{-1}{n^2} \quad (6)$$

$$(5) \Rightarrow R_n(y) = y\,n^{\frac{-1}{n^2}} = y\,e^{\left(\frac{-1}{n^2}\ln n\right)} \equiv \boxed{y\cdot \exp\left(\frac{-1}{n^2}\ln n\right)} \quad (7)$$

<u>Side Comment:</u> In many applications we confront expressions such as $e^{f(x,y,\dots)}$ where $f(x,y,\dots)$ is some complicated expression. To make it easier to write/print we widely use:

$$\boxed{e^{f(x,y,\dots)} \equiv \exp(f(x,y,\dots))} \quad (8)$$

Problem 4.3.1 (v) continued...

$$(5) \Rightarrow R_n(y) = y \cdot \exp\left(-\frac{1}{n^2}\ln n\right) \Rightarrow R(y) = \lim_{n\to\infty} y \cdot \exp\left(-\frac{1}{n^2}\ln n\right) \quad (9)$$

As $n \to \infty$ $\ln n \to \infty$ and $n^2 \to \infty \Rightarrow \left(-\frac{1}{n^2}\ln n\right) \to -\frac{\infty}{\infty}$ (10)

To give the expression in (...) meaning we apply l'Hopital's rule:

$$\lim_{n\to\infty}\left(-\frac{1}{n^2}\ln n\right) = \lim_{n\to\infty}\left(\frac{-\frac{d}{dn}\ln n}{\frac{d}{dn}n^2}\right) = \lim_{n\to\infty}\left(\frac{-\frac{1}{n}}{2n}\right) \quad (11)$$

$$= \lim_{n\to\infty}\left(\frac{-1}{2n^2}\right) = 0 \quad (12)$$

This is another example of the result we have seen previously, that the behavior of $\ln x \otimes$ (polynomial in x) is determined by the polynomial whether it is in the numerator or the denominator.

Returning to (9) $R(y) = \lim_{n\to\infty} y \cdot \exp\left(-\frac{1}{n^2}\ln n\right) = y e^0 = y$ (12)

$$|R(y)| < 1 \Rightarrow |y| < 1 \Rightarrow |1/x| < 1 \Rightarrow \boxed{|x| > 1} \quad (13)$$

Problem 4.3.1 (vii)

Show that $\sum_{1}^{\infty} n!\,x^n$ has zero radius of convergence

$$R_n = \frac{(n+1)!\,x^{n+1}}{n!\,x^n} = (n+1)x \Rightarrow R = \lim_{n\to\infty} R_n = \infty$$

Since this holds for all (finite!) values of x, this series has zero radius of convergence.

4.3.1 (vi): $\sum_{1}^{\infty} \frac{(x-1)^n}{n x^n} = \sum_{1}^{\infty} \frac{1}{n}\left(\frac{x-1}{x}\right)^n = \sum_{1}^{\infty} \frac{1}{n}\left(1-\frac{1}{x}\right)^n$

As before let $y = 1/x \Rightarrow \Sigma = \Sigma \frac{1}{n}(1-y)^n \Rightarrow$

$$R_n = \frac{1}{n+1}(1-y)^{n+1} \cdot \frac{n}{(1-y)^n} = (1-y)\frac{n}{n+1} \Rightarrow$$

$$R = \lim_{n\to\infty} R_n = 1-y = 1-\frac{1}{x} = \frac{x-1}{x} \; ; \; |R| < 1 \Rightarrow \left|\frac{x-1}{x}\right| < 1$$

$$\Rightarrow |x-1| < |x| \Rightarrow (x-1)^2 < (x)^2$$
$$\hookrightarrow x^2 - 2x + 1 < x^2 \Rightarrow -2x+1 < 0 \Rightarrow 2x-1 > 0$$
$$\Rightarrow \boxed{x > \frac{1}{2}}$$

ADDITIONAL TRICKS / SHORTCUTS

As noted previously, we often encounter a function of the form

$$f(x) = e^{g(x)} \equiv \exp(g(x)) \qquad (1)$$

which we wish to expand in a power series in the variable x. A fast way to do this is to write

$$e^{g(x)} = 1 + \frac{g(x)}{1!} + \frac{1}{2!} g^2(x) + \frac{1}{3!} g^3(x) + \cdots \qquad (2)$$

Then expand $g(x)$ in a series: $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, (3) and collect all the terms obtained by combining (2) and (3).

EXAMPLE: (text p.83) : $f(x) = \exp\left(\frac{x}{1-x}\right)$; $g(x) = \frac{x}{1-x}$ (4)

$$\Rightarrow f(x) = 1 + \frac{1}{1!}\left(\frac{x}{1-x}\right) + \frac{1}{2!}\left(\frac{x}{1-x}\right)^2 + \frac{1}{3!}\left(\frac{x}{1-x}\right)^3 + \cdots \qquad (5)$$

If $|x| < 1$ we can then expand: $\frac{x}{1-x} = x(1 + x + x^2 + x^3 + \cdots)$ (6)

Then: $f(x) = 1 + \frac{1}{1!} x(1 + x + x^2 + x^3 + \cdots) + \frac{1}{2!} x^2 (1 + x + x^2 + x^3)^2 + \cdots$ (7)

$$= 1 + x + \frac{3}{2} x^2 + \cdots \qquad (8)$$

Since we are evaluating $f(x)$ for some fixed numerical value of x, we can continue in (7) and (8) until the next power of x in (8) is too small to be of interest.

ASYMPTOTIC EXPANSIONS

The method of steepest descents (saddle point method) leads naturally into the subject of asymptotic expansions or asymptotic series:

The approximate result contained in the MASTER FORMULA on p. 130.1 can be shown to be the first term in an asymptotic expansion.

Convergent series: $f(z) = \sum_{n=0}^{N} a_n z^n$; series approaches $f(z)$ for fixed z as $N \to \infty$ (1)

Asymptotic series: $f(z) = \sum_{n=0}^{N} b_n \frac{1}{z^n}$; Series approaches $f(z)$ for fixed N as $z \to \infty$ (2)

Example: The incomplete Gamma Function: $I(x,p) = \int_x^\infty du\, e^{-u} u^{-p}$ (3)

Evidently: $I(0,p) = \int_0^\infty du\, e^{-u} u^{-p} = \Gamma(1-p)$ (4)

Integrating (3) by parts gives: $I(x,p) = -\frac{e^{-u}}{u^p}\Big|_x^\infty - \int_x^\infty (-e^{-u})(-p u^{-p-1})\,du$ (5)

$\therefore\ I(x,p) = \frac{e^{-x}}{x^p} - p\int_x^\infty du\, e^{-u} u^{-p-1}$ another application of partial integration (6)

Continuing in this manner we develop the following series:

$$I(x,p) = \frac{e^{-x}}{x^p} - p\frac{e^{-x}}{x^{p+1}} + p(p+1)\frac{e^{-x}}{x^{p+2}} - p(p+1)(p+2)\frac{e^{-x}}{x^{p+3}} + p(p+1)(p+2)(p+3)\int_x^\infty du\, e^{-u} u^{-p-4} \quad (7)$$

After many such integrations:

$$\boxed{I(x,p) = e^{-x}\left\{\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \frac{p(p+1)(p+2)}{x^{p+3}} + \dots\right\} + (-1)^n \frac{(p+n-1)!}{(p-1)!}\int_x^\infty du\, e^{-u} u^{-p-n}} \quad (8)$$

We can verify [redacted] the general expression in (8) by noting that the expression in (7) corresponds to $n=4$, so that from (8) the coefficient of the integral should be:

$$\text{Coefficient in (7)} = (-1)^4 \frac{(p+4-1)!}{(p-1)!} = \frac{+(p+3)(p+2)(p+1)p\,(p-1)!}{(p-1)!} = (p+3)(p+2)(p+1)p \quad \checkmark \quad (9)$$

We next test the series in (8) for convergence using the d'Alembert ratio test:

By inspection:

$$\boxed{\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{(p+n)!}{(p+n-1)!}\,\frac{1}{x} = \lim_{n\to\infty}\frac{p+n}{x} = \infty} \quad (10)$$

This is the key equation to understanding what an asymptotic series is:

$$\frac{p+n}{x} \begin{cases} \xrightarrow{\infty} & x\text{-fixed};\ n\to\infty \\ \xrightarrow{0} & n\text{-fixed};\ x\to\infty \end{cases} \quad (11)$$

Hence there is no value of x for which the series in (8) converges formally.

Nonetheless we can show that this series is a good numerical approximation to $I(x,p)$.

Consider the partial sum $S_n(x,p)$ defined by

$$I(x,p) \equiv S_n(x,p) + (-1)^{n+1}\frac{(p+n)!}{(p-1)!}\int_x^\infty du\, e^{-u} u^{-p-n-1} \quad \Rightarrow \quad (12)$$

$$|I(x,p) - S_n(x,p)| \le \frac{(p+n)!}{(p-1)!}\int_x^\infty du\, \underbrace{|e^{-u}|}_{\le 1}\, |u^{-p-n-1}| \le \frac{(p+n)!}{(p-1)!}\int_x^\infty du\, u^{-p-n-1}$$

this is the biggest value of e^{-u}

$$\le \frac{(p+n)!}{(p-1)!}\left|\frac{1}{u^{p+n}}\left(\frac{1}{p+n}\right)\right|_x^\infty = \frac{(p+n-1)!}{(p-1)!}\,\frac{1}{x^{p+n}} \quad (13)$$

-100

It follows from (13) that as $x \to \infty$ $|I(x,p) - S_n(x,p)| \to 0$, so that for fixed n $S_n(x,p)$ approaches the exact result $I(x,p)$. Hence such an asymptotic series is perfectly good for numerical computations, even though it does not formally converge to $I(x,p)$.

Numerical Results: Following ARFKEN we examine the case $I(x, p=1)$:

$$I(x,p=1) \equiv E_1(x) = \int_x^\infty du\, e^{-u} u^{-1} \Rightarrow e^x E_1(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots \quad (14)$$

As we will see, another difference between a convergent series and an asymptotic series is that including more terms does not necessarily give a better numerical result. Instead there is an optimum number of terms, which in this case is $n=5$. Here are the results for $x=5$:

$$S_1 = \frac{1}{x} = 0.2000 \qquad S_2 = \frac{1}{x} - \frac{1}{x^2} = \frac{1}{5} - \frac{1}{25} = 0.1600$$

$$S_3 = 0.16 + \frac{2}{125} = 0.1760 \qquad S_4 = 0.1760 - \frac{6}{625} = 0.1664$$

$$S_5 = 0.1664 + \frac{24}{5 \times 625} = 0.1741 \qquad S_6 = 0.1741 - \frac{120}{25 \times 625} = 0.1664$$

$$S_7 = 0.1664 + \frac{6!}{5^7} = 0.1756 \qquad S_8 = 0.1756 - \frac{7!}{5^8} = 0.1627$$

We see that S_n for $n=$ even are all smaller than S_n for $n=$ odd. As shown in the accompanying figure, the best numerical approximation is obtained at the point of closest approach of the even and odd S_n:

$$\boxed{0.1664 \le e^x E_1(x)\big|_{x=5} \le 0.1741} \quad (15)$$

exact value $\rightarrow$ 0.1704

Another Example of an Asymptotic Expansion:

Cosine and Sine Integrals

$$Ci(x) = -\int_x^\infty dt\, \frac{\cos t}{t} \quad ; \quad Si(x) = -\int_x^\infty dt\, \frac{\sin t}{t} \qquad (1)$$

We can also define the related functions: $\boxed{f(x) = Ci(x)\sin x - Si(x)\cos x}$ (2)

Hence:

$$f(x) = -\int_x^\infty \frac{dt}{t}\underbrace{(\sin x\cos t - \cos x\sin t)}_{\sin(x-t)} = -\int_x^\infty dt\, \frac{\sin(x-t)}{t} \qquad (3)$$

Let $y = t - x$ (for fixed x) $\Rightarrow$ $dy = dt$; $t = \infty \Rightarrow y = \infty$; $t = x \Rightarrow y = 0$

$$\boxed{\therefore f(x) = -\int_0^\infty \frac{dy}{y+x}\sin(-y) = \int_0^\infty dy\, \frac{\sin y}{y+x}} \qquad (4)$$

Similarly: $g(x) \equiv -Ci(x)\cos x - Si(x)\sin x = \int_x^\infty dt \frac{(\cos t\cos x + \sin t\sin x)}{t}$ (5)

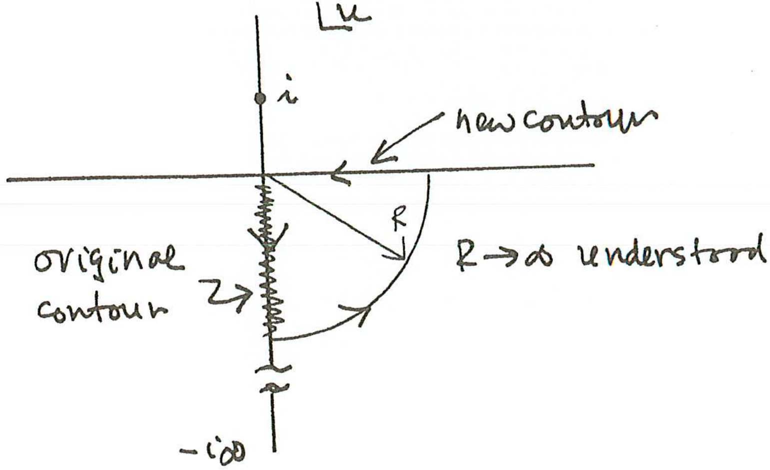
$$\boxed{\therefore g(x) = \int_x^\infty dt\, \frac{\cos(t-x)}{t} = \int_0^\infty dy\, \frac{\cos y}{y+x}} \qquad (6)$$

From (4) & (6): $g(x) + if(x) = \int_0^\infty dy \frac{(\cos y + i\sin y)}{y+x} = \int_0^\infty dy \frac{e^{iy}}{y+x}$ (7)

Let $u \equiv -iy/x$ (for fixed x) $\Rightarrow y = iux \Rightarrow dy = ix\,du$; $y = 0 \Rightarrow u = 0$
$y = \infty \Rightarrow u = -i\infty$

$$\therefore g(x) + if(x) = \int_0^{-i\infty} \frac{(ix\,du)\,e^{-ux}}{iux + x} = i\int_0^{-i\infty} \frac{du\, e^{-ux}}{1+iu} \qquad (8)$$

We wish to evaluate this integral by contour integration noting that the integrand has a singularity (simple pole) at $u = +i$.



For the contour shown $\oint_C = 0$ since the only singularity lies outside the contour.

Thus $$0 = \oint_C = \int_0^{-i\infty} + \int_{R\to\infty} + \int_\infty^0 \qquad (9)$$

As usual we argue that $\int_R \to 0$ since it is damped by the exponential factor e^{-ux}.

<u>Note</u>: This argument applies because no part of $\int_R$ gets near the origin as $R\to\infty$; However the same is not true for the other two contributions in (9). Hence (9) ⇒

$$\oint = 0 = \int_0^{-i\infty} + 0 + \int_\infty^0 \Rightarrow \int_0^{-i\infty} = -\int_\infty^0 = +\int_0^\infty \qquad (10)$$

Hence in (8): $$g(x) + i f(x) = i\int_0^\infty du \frac{e^{-ux}}{1+iu} = \int_0^\infty du \frac{(u+i)\,e^{-ux}}{1+u^2} \qquad (11)$$

Equating real and imaginary parts in (11) ⇒

$$g(x) = \int_0^\infty du \frac{u\,e^{-ux}}{1+u^2} \; ; \qquad f(x) = \int_0^\infty du \frac{e^{-ux}}{1+u^2} \qquad (12)$$

For these integrals to converge we must have $\mathrm{Re}\, x > 0$ so that the exponential is damped. [Also recall that from (7) & (8) $f(x)$ & $g(x)$ are real.]

We can evaluate (12) as asymptotic expansions by defining

$$ux = v \Rightarrow x\,du = dv \qquad (13)$$

Then
$$\boxed{g(x) = \frac{1}{x^2}\int_0^\infty dv\, \frac{v e^{-v}}{1+v^2/x^2} \quad ; \quad f(x) = \frac{1}{x}\int_0^\infty dv\, \frac{e^{-v}}{1+v^2/x^2}} \tag{14}$$

KEY POINT!!! Here is where the asymptotic expansion enters!

We wish to expand the denominators so as to evaluate the integrals by an infinite series:
$$\frac{1}{1+W} = 1 - W + W^2 - W^3 + \cdots = \sum_{n=0}^\infty (-1)^n W^n \quad ; \quad \boxed{W = v^2/x^2} \tag{15}$$

However, such an expansion only makes sense when $W = v^2/x^2 < 1$. The problem is that whatever the (finite) value of x is, v^2/x^2 will be >1 at some point in the integration, since $0 \le v \le \infty$. Hence expanding as in (15) does not seem to make sense mathematically. However, for x sufficiently large, $v^2/x^2 > 1$ will only occur for values of v that are sufficiently large that they make a negligible contribution to the integral, due to the damping factor e^{-v}. Hence for sufficiently large x, we can use (15) in (14) knowing that the numerical error will be negligible. This is specifically where the concept of an asymptotic expansion enters. Combining (14) & (15) we then have:

$$f(x) \cong \frac{1}{x}\int_0^\infty dv\, e^{-v} \sum_{n=0}^\infty (-1)^n \frac{v^{2n}}{x^{2n}} \quad ; \quad g(x) \cong \frac{1}{x^2}\int_0^\infty dv\, e^{-v} \cdot v \sum_{n=0}^\infty (-1)^n \frac{v^{2n}}{x^{2n}} \tag{16}$$

We can evaluate these integrals term-by-term using the integral representation of $\Gamma(n+1)$:
$$\Gamma(n+1) = n! = \int_0^\infty dt\, e^{-t} t^n \quad \Rightarrow \tag{17}$$

$$\underbrace{\int_0^\infty dv\, e^{-v} v^{2n} = (2n)!}_{\text{use in } f(x)} \quad ; \quad \underbrace{\int_0^\infty dv\, e^{-v} v^{2n+1} = (2n+1)!}_{\text{use in } g(x)} \tag{18}$$

Combining (16) & (18):

$$\boxed{f(x) \cong \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} \quad ; \quad g(x) \cong \frac{1}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}} \quad (19)$$

for large x

We can now invert Eqs. (3) & (5) above to solve for the original functions $C_i(x), S_i(x)$.

$$\cos x \cdot f(x) + \sin x \cdot g(x) = \left\{ \cancel{C_i(x)\cos x \sin x} - S_i(x)\cos^2 x \right\} + \left\{ \cancel{-C_i(x)\sin x \cos x} - S_i(x)\sin^2 x \right\} \quad (20)$$

$$\therefore \cos x \cdot f(x) + \sin x \cdot g(x) = -S_i(x)\left[\cos^2 x + \sin^2 x\right] = -S_i(x)$$

Similarly:

$$\boxed{\begin{aligned} S_i(x) &= -\cos x \cdot f(x) - \sin x \cdot g(x) \\ C_i(x) &= \sin x \cdot f(x) - \cos x \cdot g(x) \end{aligned}} \quad (21)$$

Hence finally, combining (19) & (21):

$$\boxed{\begin{aligned} C_i(x) &\cong \frac{\sin x}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\cos x}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}} \\ S_i(x) &= -\frac{\cos x}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\sin x}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}} \end{aligned}}$$

.4. Summary

Here is the summary of this chapter.

- The series $\sum_{n=0}^{\infty} a_n$ converges to S if $|S - S_N| < \varepsilon$ for $N > N(\varepsilon)$ where S_N is the sum of the first N terms.

- The series
$$\sum_{n=0}^{\infty} a_n$$
converges absolutely if the series with a_n replaced by $|a_n|$ converges. If a series converges but not if the terms are replaced by absolute values, then it converges conditionally.

- Know the following about the geometric series
$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \qquad |r| < 1$$

- The series
$$\sum_{n=0}^{\infty} a_n$$
of positive terms converges if $a_n \le b_n$ for n beyond some value and if the sum over the positive numbers b_n converges. This is called the comparison test.

- The series
$$\sum_{n=0}^{\infty} a_n$$
converges absolutely if
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$
This is the ratio test.

- The sum of positive monotonic terms $\sum_{n=0}^{\infty} f(n)$ converges or diverges along with $\int^L f(x)dx$ as $L \to \infty$. To do the integral it is enough to use an integrand that agrees with f for large x. You can also trade the given f for another smaller one (while showing the sum diverges) or a larger one (to show it converges) if that makes it easy to integrate.

- The power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely within the interval of convergence $|x| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| .$$

- Relations between functions will be satisfied order by order when they are replaced by their power series. You must know how to expand functions of functions, out to some desired order within the common interval of convergence.

- The power series representing a function may be integrated term by term and differentiated term by term within the interval of convergence to obtain the series for the integral or derivative of the function in question.

ASYMPTOTIC SERIES

A series of the form $f(x) = \sum_{n=0}^{N} b_n \frac{1}{x^n}$

which approaches $f(x)$ for fixed N as $x \to \infty$.

Such a series may not be convergent, but is nonetheless a valid approximation to $f(x)$.

This contrasts to a convergent series for which

$$f(x) = \sum_{n=0}^{N} a_n x^n$$

which converges to $f(x)$ for fixed x as $N \to \infty$.