# VECTOR SPACES

A Vector Space is defined over a field F: See below

Then a vectors space is a set of elements $\{V\}$ and an operation + such that

a) $\{V, +\}$ = Abelian group

Also:

b) If $\beta, \alpha \in F$ and $x \in V$ then $\alpha x$ is also in $V$, and

b1) $\alpha(\beta x) = (\alpha\beta)x$

b2) $1x = x \quad \forall x$

c) distributivity

c1) $\alpha(x+y) = \alpha x + \alpha y$

c2) $(\alpha+\beta)x = \alpha x + \beta x$

Terminology: F = real numbers ⇒ Real Vector Space

F = complex " ⇒ Complex " "

Examples of Vector Spaces: (Prototype Vector space) ?

1) Set of all n-tuples: $x = (x_1, x_2, x_3, \ldots, x_n)$

$\alpha x = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$ ; $x+y = (x_1+y_1, \cdots x_n+y_n)$

$0 = (0, 0, \cdots 0)$

2) Set of all complex numbers

The numbers themselves can be viewed as vectors

$$X = a + ib$$

3) The set of all polynomials in a variable $t$

$$X_1 = P_1(t) = a_1 + b_1 t + c_1 t^2 + \cdots$$
$$X_2 = P_2(t) = a_2 + b_2 t + c_2 t^2 + \cdots$$

This space is infinite-dimensional, as we discuss later when we develop the theory of Hilbert Space

## LINEAR INDEPENDENCE

(l.i.)

A finite set of vectors $\{x_i\}$ is linearly-independent if and only if

$$\boxed{\sum_i \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \ \forall i}$$

To understand this suppose that $a_j \neq 0$. Then one can solve for $x_j$:

$$x_j = -\frac{1}{a_j}(\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n)$$

This expresses $x_j$ in terms of the other vectors, so $x_j$ is not linearly independent. (Note that the vector $X=0$ is linearly-dependent since one can express it in terms of any other vector $x$: $0 = 0 \cdot x$)

EXAMPLES: (1) $x_1(t) = 1-t$ $\quad x_2(t) = t - t^2$ $\quad x_3(t) = 1 - t^2$

These 3 vectors are linearly-dependent since

$$\sum_{i=1}^{3} \alpha_i x_i(t) = 0$$

can be satisfied with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$, so $\alpha_i \neq 0$.

(2) $x_0(t) = 1$ $\quad x_1(t) = t$, $\ldots$ $x_n(t) = t^n$

Form $\sum_{i=1}^{n} \alpha_i x_i(t) = 0$

In this case the only way that this equation can hold is if all $\alpha_i \equiv 0$ [consequence of a theorem of algebra]

## BASES AND DIMENSIONALITY

A basis in $V$ is a set $\{x_i\}$ for $x_i \in V$ such that every $x \in V$ is a linear combination of the $x_i$:

$$\boxed{x = \sum_{i=1}^{n} \alpha_i x_i} \qquad \forall x$$

If the $\{x_i\}$ are finite in number then $V$ is a finite-dimensional vector space. Otherwise it is infinite-dimensional

Examples:

1) In 3-dim $\{x_i\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ where

$\hat{\imath} = (1,0,0)$ $\quad \hat{\jmath} = (0,1,0)$ $\quad \hat{k} = (0,0,1)$

2) For the polynomials the basis can be taken to be $\{x_n(t)\}$ where $x_n(t) = t^n$. This is an infinite basis. We will later see that this is the starting point of our discussion of <u>HILBERT SPACE</u>

<u>Theorem:</u> For $\forall\, x \in V$ there is a <u>unique</u> representation in terms of a given basis $\{x_i\}$.

<u>Proof:</u> Assume the contrary, then

$$x = \sum_i \alpha_i x_i \quad \text{and also} \quad x = \sum_i \beta_i x_i$$

Then $$0 = \sum_i (\alpha_i - \beta_i) x_i \equiv \sum_i \gamma_i x_i$$

But the $\{x_i\}$ are linearly independent (by assumption)

$\Rightarrow \gamma_i \equiv 0 \Rightarrow \boxed{\alpha_i = \beta_i} \leftarrow$ uniqueness

<u>Definition:</u> <u>Dimension</u> of a Vector space = (unique) number of basis Vectors

<u>ISOMORPHISM OF VECTOR SPACES</u>: ("Iso" = SAME ; "MORPH" = FORM)

The space spanned by the n-tuples in n-dimensions $\equiv F^n$

Ex: $F^3$ is spanned by $(1,0,0)$ $(0,1,0)$ $(0,0,1)$.

We can then show that every finite n-dimensional vector space is isomorphic to $F^n$.

Definition: Two vector spaces (U and V) are <u>isomorphic</u> if

a) U and V are over the same field

b) There exists a 1-1 correspondence between $x \in U$ and $y \in V$ such that $y = T(x)$ where

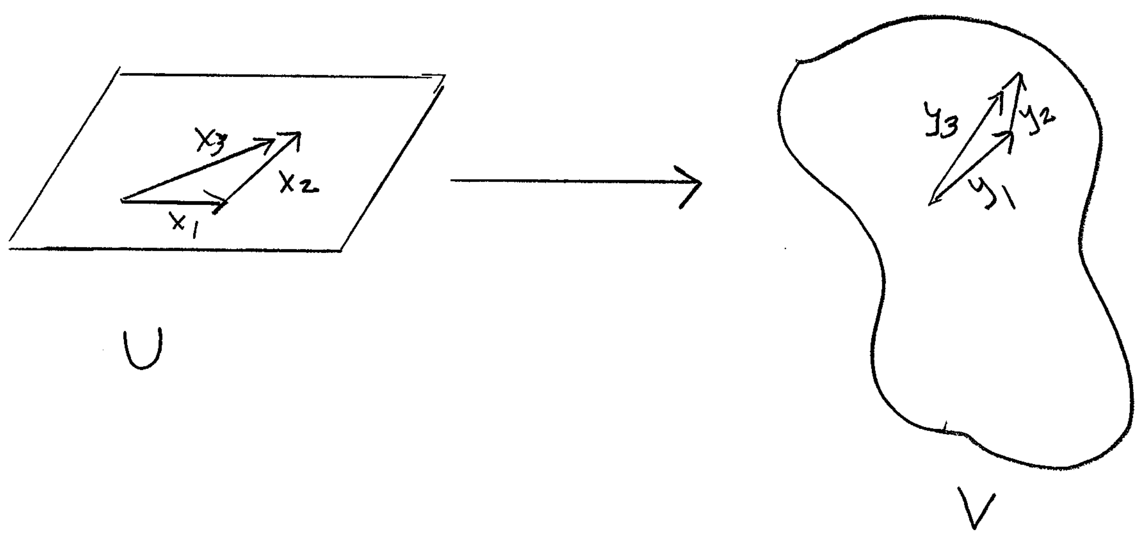
$$T(x_1 + x_2) = T(x_1) + T(x_2) = y_1 + y_2$$
$$T(\alpha x_1) = \alpha T(x_1) = \alpha y_1$$

These relations can be grouped into a single equation

$$\boxed{T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) = \alpha_1 y_1 + \alpha_2 y_2}$$

"preserves algebraic structure"

Schematically this can be visualized as follows:




Simply stated: If $x_1 \to y_1$, and $x_2 \to y_2$, and $x_3 \to y_3$, then if the effect of adding $x_1$ and $x_2$ in U is to give $x_3$, then it must be the case also that the effect of adding $y_1$ and $y_2$ in V gives the corresponding vector $y_3$.

# ISOMORPHISMS MORE GENERALLY

Consider 2 systems $S = \{E, *\}$ and $S' = \{E', \star\}$

We wish to establish an isomorphism between $S$ & $S'$:

| $E$ | $f$: 1-1 mapping | $E'$ |
| --- | --- | --- |
| $a$ | $\longrightarrow$ | $f(a) \equiv a'$ |
| $b$ | $\longrightarrow$ | $f(b) \equiv b'$ |
| $a * b = c$ | $\longrightarrow$ | $f(a) \star f(b) \stackrel{?}{=} f(c) = f(a*b)$<br>$a' \star b' \stackrel{?}{=} c'$ |
| $c$ | $\longrightarrow$ | $f(c) = c'$ |

If the ? turns out correctly then $S$ and $S'$ are <u>isomorphic</u>.

The two systems $S$ and $S'$ can be isomorphic even if the operations $*$ and $\star$ have nothing to do with each other. This is illustrated by the following examples where the elements of $S$ are rotations, whereas the elements of $S'$ are permutations:

isomorphic systems that at first glance look very different. Consider the six 2x2 matrices:

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$$\underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \qquad \underset{A}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \qquad \underset{B}{\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}}$$

$$\underset{C}{\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}} \qquad \underset{D}{\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}} \qquad \underset{E}{\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}}.$$

$\begin{vmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{vmatrix}$

These six matrices form a group which can be verified. We give the multiplication table for this group which tells us everything about it. Note that this group is not Abelian.

| | I | A | B | C | D | E |
|---|---|---|---|---|---|---|
| I | I | A | B | C | D | E |
| A | A | I | D | E | B | C |
| B | B | E | I | D | C | A |
| C | C | D | E | I | A | B |
| D | D | C | A | B | E | I |
| E | E | B | C | A | I | D |

AB = D

First, it is clear that the system is closed. The unit element of the group is the matrix I. To determine the inverse of a given element (say C) we look at the multiplication table to see what element X is such that CX = XC = I. Answer $C^{-1} = C$. Also, $D^{-1} = E$. Matrix multiplication is associative (to be proven later), so we have a group. Let us refer to this system as $\{M, \cdot\}$, M for the set of six matrices and $\cdot$ for matrix multiplication.

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Now we consider the set of six permutations of three numbers as another system:

$$\underset{I'}{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}} \qquad \underset{A'}{\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}} \qquad \underset{B'}{\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}}$$

$$\underset{C'}{\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}} \qquad \underset{D'}{\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}} \qquad \underset{E'}{\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}}$$

The product of two permutations is the single permutation that accomplishes what two applied successively would accomplish, e.g.

i.e. the map that $1 \to 3$, $2 \to 1$, $3 \to 2$

1) $$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} \odot \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$$
$$B' \odot D' = C'$$

2) $$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} \odot \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$$
$$A' \odot B' = D'$$

E

3) $$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} \odot \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$$
$$I' \odot C' = C'$$

4) $$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} \odot \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$$
$$E' \odot C' = A'$$

5) $$\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} \odot \begin{pmatrix}1 & 2 & 3\\ x & y & z\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ z & x & y\end{pmatrix}$$

# ANGULAR MOMENTUM AND ISOMORPHISMS



A common example of the application of the ideas of isomorphisms is the theory of angular momentum in quantum mechanics. The angular momentum operator $\vec{J}$ for spin-1/2 particles is given by

KNOW THESE!!

$$\vec{J} = \frac{\hbar}{2}\vec{\sigma} \tag{1}$$

$$\boxed{\sigma_x \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \tag{2}$$

By direct computation we can establish that these operators satisfy the commutation relations

$$\boxed{[J_x, J_y] = i\hbar J_z \;, \; \underline{\text{et. cycl.}}} \tag{3}$$

$|\vec{J}| = \frac{1}{2}\hbar$ is the smallest angular momentum that any object can have, and the representation of $\vec{J}$ as in (1) & (2) is called the <u>FUNDAMENTAL REPRESENTATION</u>, and this defines the algebra of angular momentum operators as in (3).

However, once the abstract relations in (3) are obtained, they can be realized by other operators: 3X3, 4X4, ... matrices, which are then said to be isomorphic to $\frac{\hbar}{2}\vec{\sigma}$. Moreover, the commutation relations in (3) can be realized by other structures, such as differential operators; these are then said to be isomorphic to the matrices in (2):

$$J_x = \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right); \; J_y = \left(z\frac{\hbar}{i}\frac{\partial}{\partial x} - x\frac{\hbar}{i}\frac{\partial}{\partial z}\right); \; J_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

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**Theorem:**

Every n-dimensional vector space $V$ over a field $F$ is isomorphic to the space $F^n$ of the n-tuples of $F$.

**Proof:** Let $\{x_1, \dots x_n\}$ be any basis in $V$. Hence for $x \in V$

$$x = \sum_i \alpha_i x_i \qquad (\alpha_i \text{ are unique}) \qquad (1)$$

The proposed isomorphism is

$$V \longleftrightarrow F^n \qquad (2)$$

$$x \in V \longleftrightarrow (\alpha_1, \alpha_2, \dots \alpha_n) \qquad (3)$$

Symbolically $\boxed{T(x) = (\alpha_1, \alpha_2, \dots \alpha_n)}$ (4)

[$T(x)$ picks out the coefficients of $x_i$ in $x = \sum_i \alpha_i x_i$]*

(1)-(4) $\Rightarrow$ If $y = \sum_i \beta_i x_i$ then $(cx+dy) = c\sum_i \alpha_i x_i + d\sum_i \beta_i x_i$

$$= \sum_i (c\alpha_i + d\beta_i)x_i \qquad (4)$$

*Since $T(\dots)$ picks out the coef. of $x_i \Rightarrow$

$$T(cx+dy) = c\alpha_i + d\beta_i = (c\alpha_1 + d\beta_1, c\alpha_2 + d\beta_2, \dots) \Leftarrow \text{OK} \qquad (5)$$

To establish an isomorphism we want to show that this is the same as

$cT(x) + dT(y)$: $\quad cT(x) = c(\alpha_1, \alpha_2, \dots \alpha_n) = (c\alpha_1, c\alpha_2, \dots c\alpha_n)$

$\quad dT(y) = d(\beta_1, \beta_2, \dots \beta_n) = (d\beta_1, d\beta_2, \dots d\beta_n)$

$\therefore cT(x) + dT(y) = (c\alpha_1 + d\beta_1, c\alpha_2 + d\beta_2, \dots c\alpha_n + d\beta_n) \Leftarrow$ OK QED

**Corollary:** Any two n-dim vector spaces over the same $F$ are isomorphic to each other [Since they are both isomorphic to $F^n$].

# LINEAR TRANSFORMATIONS (l.t.)

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A l.t. $\underline{\underline{A}}$ on $V$ assigns to every $x \in V$ another vector $\underline{\underline{A}}x \in V$ such that

$$\boxed{A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)} \qquad (1)$$

The product: $Px = (AB)x \equiv A(Bx)$ (2)

Examples:

1) Rotations in a ~~plane~~ plane: $A \equiv R =$ Rotation operator (3)

$$\vec{x} \to R\vec{x} = R(\theta)\vec{x}$$

2) (4)

$$0x = 0$$
$$Ix = x$$

3) $D =$ differentiation $\Rightarrow Df(x) \equiv \frac{df}{dx}$ (5)

$X =$ multiplication by $x \Rightarrow Xf(x) = xf(x)$

Then: $(DX)f(x) = D(Xf(x)) = \frac{d}{dx}(xf(x)) = f(x) + x\frac{df(x)}{dx}$ (6)

$(XD)f(x) = X\left(\frac{df(x)}{dx}\right) = x\frac{df(x)}{dx}$ (7)

$\therefore [DX - XD]f(x) = f(x) \Leftrightarrow \boxed{[D, X] = I}$ (8)

This commutation relation is at the heart of quantum mechanics!

Let $p_x \to -i\hbar\frac{d}{dx} = -i\hbar D$. Then (8) $\Rightarrow$ $\boxed{[X, p_x] = i\hbar}$

4) In the space of polynomials of degree $m$, $P_m$ define the linear transformation $D$ via

$$D P_m(x) = \frac{d}{dx}P_m(x)$$

$$D^1 \equiv D, \quad D^2 \equiv DD \quad ; D^n = \underbrace{DD\cdots}_{n \text{ factors}}$$