Physical Interpretation of Bessel's Inequality:

Consider a 3-dim vector $V = \sum_{i=1}^{3} \alpha_i x_i$ $\qquad \alpha_i = \langle x_i | V \rangle = V_i$

$x_i = \hat{\imath}, \hat{\jmath}, \hat{k}$

Then we can also write $V = \sum_{i=1}^{3} V_i x_i$ (9)

$\rightarrow |\vec{V}|^2 = \sum_{i=1}^{3} |V_i|^2$

this holds if $\{x_i\}$ is a complete orthonormal (CON) basis.

Suppose now that $\{x_i\}$ are orthonormal, but not complete; Ex:
$\{x_i\} = \{\hat{\imath}, \hat{\jmath}\}$ (but not $\hat{k}$) in 3-dim. Then:

$$\sum_{i=1}^{2} |V_i|^2 = V_x^2 + V_y^2 = |\vec{V}|^2 - V_z^2 \Rightarrow \boxed{\sum_{i=1}^{2} |V_i|^2 \leq |\vec{V}|^2} \quad (10)$$

The equality would hold iff $V_z \equiv 0$.

Continuing this example, the second part of Bessel's inequality states that the vector $x' = x - \sum_j \alpha_j x_j$ is orthogonal to the other x_j's

In this example

$$\vec{x}' = \vec{V} - \sum_{i=1}^{2} V_i x_i = \vec{V} - V_x \hat{\imath} - V_y \hat{\jmath} \equiv V_3 \hat{k}$$

Obviously this vector is orthogonal to $V_x \hat{\imath} + V_y \hat{\jmath}$. ✓

Completeness:

This is a crucial concept. The following statements are equivalent descriptions of completeness:

Let $X = \{x_i\}$ be a set of m orthonormal vectors in a finite vector space V.

(1) X is complete

(2) If $\langle x_i | x \rangle = 0$ for $i = 1, \dots, m \Rightarrow x = 0$

(3) X spans V

(4) If $x \in V$ then $x = \sum_i \langle x_i | x \rangle x_i$

(5) If $x, y \in V$ then $\boxed{\langle y | x \rangle = \sum_i \langle y | x_i \rangle \langle x_i | x \rangle}$ (1)

Parseval's Identity

(6) If $x \in V$ then $|x|^2 = \sum_i |\langle x_i | x \rangle|^2 = \sum_i |\alpha_i|^2$ (2)

Proof: We show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2) [We show not(2) $\Rightarrow$ not(1)]: If $\langle x_i | x \rangle = 0$ but $x \neq 0$ than we can adjoin $x/|x|$ to X to form a larger set X' bigger than X, so X is not complete.

(2) $\Rightarrow$ (3) If X does not span V $\Rightarrow$ $\exists$ an x which cannot be written as

$$x = \sum_i \alpha_i x_i = \sum_i \langle x_i | x \rangle x_i \Rightarrow x' = x - \sum_i \langle x_i | x \rangle x_i \neq 0 \qquad (3)$$

However, we can still show that $\langle x_j | x' \rangle = 0$ even though $x' \neq 0$ (4)

$$\langle x_j | x' \rangle = \langle x_j | x \rangle - \sum_i \langle x_i | x \rangle \underbrace{\langle x_j | x_i \rangle}_{\delta_{ji}} = \langle x_j | x \rangle - \langle x_j | x \rangle = 0 \qquad (5)$$

$\therefore$ not(3) $\Rightarrow$ not(2)

(3) ⇒ (4): If every x has the form $x = \sum_j \alpha_j x_j$ (with α_j not yet specified) then form

$$\langle x_i | x \rangle = \sum_j \alpha_j \underbrace{\langle x_i | x_j \rangle}_{\delta_{ij}} = \alpha_i \longrightarrow \boxed{x = \sum_i \langle x_i | x \rangle x_i} \quad (6)$$

(4) ⇒ (5): If $x = \sum_i \alpha_i x_i$ & $y = \sum_j \beta_j x_j$; $\alpha_i = \langle x_i | x \rangle$ $\beta_j = \langle x_j | y \rangle$ (7)

Hence $$\langle y | x \rangle = \langle \sum_j \beta_j x_j \,|\, \sum_i \alpha_i x_i \rangle = \sum_{i,j} \beta_j^* \alpha_i \underbrace{\langle x_j | x_i \rangle}_{\delta_{ji}} = \sum_i \beta_i^* \alpha_i \quad (8)$$

$$\therefore \quad \boxed{\langle y | x \rangle = \sum_i \langle y | x_i \rangle \langle x_i | x \rangle} \quad (9)$$

← "inserting complete set of states"
VERY USEFUL IN QM!!

PARSEVAL'S IDENTITY

(5) ⇒ (6): Set $y = x$ in PARSEVAL: $\langle x | x \rangle = |x|^2 = \sum_i \underbrace{\langle x | x_i \rangle}_{\alpha_i^*} \underbrace{\langle x_i | x \rangle}_{\alpha_i} = \sum_i |\alpha_i|^2$ (10)

(6) ⇒ (1): If X was not complete and was therefore contained in a larger set $X' = \{x_i, x_0\}$ where $\langle x_0 | x_i \rangle = 0$. Then

$$|x_0|^2 = \sum_i |\langle x_i | x_0 \rangle|^2 = 0 \Rightarrow x_0 = 0$$

Hence the only vector not in X which is orthogonal to all the x_i is the zero vector (which is trivial).

This completes the proof (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (1).

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SCHWARZ'S INEQUALITY:

[This is a generalization of $|\cos\theta| \leq 1$]

Theorem: For $x, y, \in V$ then $\boxed{|\langle x|y\rangle| \leq \sqrt{\langle x|x\rangle\langle y|y\rangle} = |x||y|}$

Proof: if $y=0$ both sides vanish ✓

if $y \neq 0$ then the set consisting of $y/|y|$ by itself is orthonormal and therefore satisfies Bessel's inequality:

$$|\langle x|y\rangle|^2 \leq |x|^2|y|^2 \Rightarrow$$

$$\| $$

$$|\langle x|\tfrac{y}{|y|}\rangle|^2 \leq |x|^2 \quad \text{or} \quad |\alpha|^2 \leq |x|^2 \quad \text{Q.E.D}$$

EXAMPLES:

1) Euclidean Space $\Rightarrow |\cos\theta| \leq 1$

2) Unitary Space $C^n \Rightarrow$ CAUCHY INEQUALITY:

for $(\alpha_1, \ldots, \alpha_n)$ & $(\beta_1, \ldots \beta_n)$

$$\boxed{\Rightarrow \left|\sum_{i=1}^{n} \alpha_i^* \beta_i\right|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \cdot \sum_{j=1}^{n} |\beta_j|^2}$$

3) In the space of polynomials:

$$\left|\int_0^1 dt\, x^*(t)y(t)\right|^2 \leq \int_0^1 dt\, |x(t)|^2 \cdot \int_0^1 dt\, |y(t)|^2$$

COMPLETE ORTHONORMAL SETS:

For convenience we want a starting point in calculations where our basis vectors are not only complete, but form a complete orthonormal (CON) set. The formal method for converting a complete set to a CON set is the GRAM-SCHMIDT method:

$$\text{Complete} \longrightarrow \text{CON}$$

Let $X = \{x_1, \dots, x_n\}$ be any basis in $\mathcal{V}$ (hence complete). We want to use these to form a $Y = \{y_1, \dots, y_n\} \ni \langle y_i | y_j \rangle = \delta_{ij}$ (1)

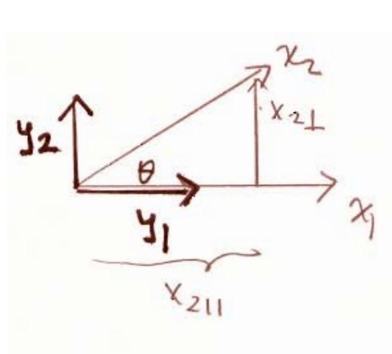
Start: ① use $x_1 \rightarrow \boxed{y_1 = x_1/|x_1|}$ (2)

② Next take x_2 and form $y_2 = (x_2 - \alpha_1 y_1)/|x_2 - \alpha_1 y_1|$ (3)

We determine $\alpha_1 \ni \langle y_1 | y_2 \rangle = 0 = \dfrac{\langle y_1 | x_2 \rangle - \alpha_1}{|x_2 - \alpha_1 y_1|} \Rightarrow \alpha_1 = \langle y_1 | x_2 \rangle$ (4)

Hence: $$\boxed{y_2 = \frac{x_2 - y_1 \langle y_1 | x_2 \rangle}{|x_2 - y_1 \langle y_1 | x_2 \rangle|}} \quad (5)^*$$

Pictorially:



$$\vec{y}_2 = \frac{\vec{x}_2 - \vec{x}_{2\parallel}}{|\vec{x}_2 - \vec{x}_{2\parallel}|} = \frac{\vec{x}_{2\perp}}{|\vec{x}_{2\perp}|} \quad (6)$$

$$\vec{x}_{2\parallel} = \hat{y}_1 |\vec{x}_{2\parallel}| = \hat{y}_1 |\vec{x}_2| \underbrace{\cos\theta}_{\hat{x}_2 \cdot \hat{y}_1} = \hat{y}_1 \underbrace{|\vec{x}_2| \hat{x}_2}_{\vec{x}_2} \cdot \hat{y}_1 = \hat{y}_1 \, \vec{x}_2 \cdot \hat{y}_1 \quad (7)$$

$$\therefore (6) \,\&\, (7) \Rightarrow \boxed{\vec{y}_2 = \frac{\vec{x}_2 - \vec{x}_2 \cdot \hat{y}_1 \hat{y}_1}{|\vec{x}_2 - \vec{x} \cdot \hat{y}_1 \hat{y}_1|}}^{\,*} \quad (b)$$

GRAM-SCHMIDT (continued)

This procedure can be repeated: At each step we start with one of the x_k and form the corresponding y_k by subtracting off the components of x_k that lie along the directions of the previously computed y_k.

Suppose that y_j vectors have been constructed, with $j = 1, 2, \ldots, r$ then

$$\boxed{|y_{r+1}\rangle = \frac{|x_{r+1}\rangle - \sum_{i=1}^{r} |y_i\rangle\langle y_i|x_{r+1}\rangle}{\left|\, |x_{r+1}\rangle - \sum_{i=1}^{r} |y_i\rangle\langle y_i|x_{r+1}\rangle \,\right|}} \quad \text{GRAM-SCHMIDT} \quad (7)$$

Check: We can verify that $|y_{r+1}\rangle$ so constructed is indeed $\perp$ to the y_k $(k \le r)$: [Dropping the normalization]

$$\langle y_k|y_{r+1}\rangle = \langle y_k|x_{r+1}\rangle - \sum_{i=1}^{r} \underbrace{\langle y_k|y_i\rangle}_{\delta_{ki}}\langle y_i|x_{r+1}\rangle \quad (8)$$

$$= \langle y_k|x_{r+1}\rangle - \langle y_k|x_{r+1}\rangle = 0 \quad \checkmark$$

CONVENIENCE OF CON BASES

Recall: $$A x_i = \sum_k \alpha_{ki} \overset{\sqrt{}\ \text{CON}}{x_k} \Rightarrow \langle x_i|A x_j\rangle = \langle x_i|\sum_k \alpha_{kj} x_k\rangle$$

$$= \sum_k \alpha_{kj} \underbrace{\langle x_i|x_k\rangle}_{\delta_{ik}\ [\text{CON}]} = \alpha_{ij} \quad (9)$$

$$\boxed{\begin{aligned} \therefore \alpha_{ij} &= \langle x_i|A x_j\rangle \\ &\equiv \langle x_i|A|x_j\rangle \end{aligned}} \quad (10)$$

$\rightarrow$ Useful in QM

SELF-ADJOINT TRANSFORMATIONS

$\hookrightarrow$ These are important in QM

Def: Adjoint Transformation: given a l.t. A on V, we define the operator $A^\dagger$ by:

$$\boxed{\langle Ax|y\rangle \equiv \langle x|A^\dagger y\rangle} \quad (1)$$

equivalent

Alternatively one can write: $\langle Ay|x\rangle^* = \langle x|Ay\rangle$ (2)

$$\langle y|A^\dagger x\rangle^* = \langle A^\dagger x|y\rangle$$

$$\boxed{\therefore \langle x|Ay\rangle = \langle A^\dagger x|y\rangle} \quad (3)$$

Def: A is self-adjoint if $A = A^\dagger$.

real V $\rightarrow$ Symmetric

Complex $\rightarrow$ Hermitian

Properties of the Adjoint

1) $A = A^\dagger$ & $B = B^\dagger \Rightarrow (A+B) = (A+B)^\dagger$

2) $A = A^\dagger$ & $\alpha \gtrless 0 \Rightarrow (\alpha A) = (\alpha A)^\dagger$ only if α = real

☆ 3) $(A)_{ij} = \alpha_{ij} \Rightarrow (A^\dagger)_{ij} = \alpha^*_{ji}$

Proof: $\alpha_{ij} = \langle x_i|Ax_j\rangle = \langle A^\dagger x_i|x_j\rangle = \langle x_j|A^\dagger|x_i\rangle^* = [(A^\dagger)_{ji}]^*$

$$\therefore (A)_{ij} = \alpha_{ij} \Leftrightarrow \alpha_{ij} = (A^\dagger)_{ji}^* \Rightarrow \alpha^*_{ij} = (A^\dagger)_{ji}$$

$$\Rightarrow (A^\dagger)_{ij} = \alpha^*_{ji}$$

If A is real $\Rightarrow A^\dagger = A^T$.