Hence $$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(\frac{-1}{2i}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0} + \frac{i}{2}\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{=0} = 0 \qquad \text{Q.E.D}$$

$$0 \leftarrow \text{C-R} \rightarrow 0$$

Returning to the previous example we could have guessed the form of $f(z)$ by noting that along the real axis where $y = 0$ we have

$$f = u(x,y) + iv(x,y) = [2x - x^3 + 3xy^2] + i[2y - 3x^2y + y^3 + \text{const}]$$

$$\longrightarrow [2x - x^3] + i[0] + \text{const}$$

This is the same expression as would have been obtained from $f(z) = 2z - z^3 + C$ along the real axis, which gives the previous answer.

2-DIMENSIONAL ELECTROSTATICS

Some electrostatics problems have a 2-dimensional geometry (with a symmetry in the 3rd dimension), that lend themselves to the use of complex variables.

Consider a charge-free region of space with some conductors. The electrostatic potential $\varphi(\vec{x})$ is constant on these surfaces [since otherwise we would have $\vec{E} = -\vec{\nabla}\varphi \neq 0 \Rightarrow$ flow of charge in a static situation].

Then

$$\vec{E} = -\vec{\nabla}\varphi \quad ; \quad \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \nabla^2\varphi = 0 \qquad (1)$$

Write $0 = \vec{\nabla}\cdot\vec{E} = \partial E_x/\partial x + \partial E_y/\partial y \Rightarrow$

$$\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y} \qquad (2)$$

Additionally: Since $\vec{\nabla}\cdot\vec{E} = 0$ in this circumstance $\vec{E}$ & $\vec{B}$ behave somewhat similarly so that we can write $\vec{E} = \vec{\nabla}\times\vec{A}$ where $\vec{A}$ is an appropriate potential.

This gives

$$E_x = \partial_y A_z - \partial_z A_y \quad ; \quad E_y = \partial_z A_x - \partial_x A_z \quad ; \quad E_z = \partial_x A_y - \partial_y A_z \qquad (3)$$

To generate a 2-dim field so that $E_z = 0 \Rightarrow A_y = A_x = \text{const} = 0$. Then $\vec{E}(x,y)$ and $\vec{A}(x,y)$ are given by

$$\underset{-\partial_x\varphi}{\underset{\parallel}{E_x}} = \partial_y A_z \Rightarrow -\frac{\partial\varphi}{\partial x} = \frac{\partial A}{\partial y} \quad ; \quad \text{Also } \underset{-\partial_y\varphi}{\underset{\parallel}{E_y}} = -\partial_x A_z \quad ; \quad \vec{A} = \hat{k}A = \hat{k}A_z \qquad (4)$$

So altogether:

$$\boxed{-\frac{\partial\varphi}{\partial x} = \frac{\partial A}{\partial y} \quad ; \quad \frac{\partial\varphi}{\partial y} = \frac{\partial A}{\partial x}} \qquad (5)$$

It follows that if we define an analytic function $f(z)$ such that

$$\boxed{f(z) = \varphi(x,y) - iA(x,y)} \qquad (6)$$

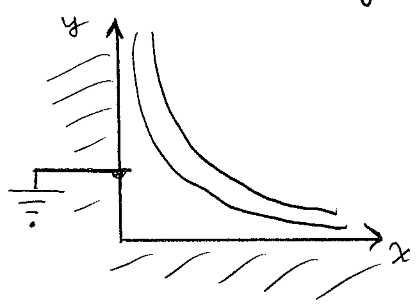
Then (5) are the C-R conditions for the complex potential.

Since $f(z)$ is analytic we can compute its derivative along any path:

$$\frac{df(z)}{dz} = \underbrace{\frac{\partial \varphi(x,y)}{\partial x}}_{-E_x} - i\underbrace{\frac{\partial A(x,y)}{\partial x}}_{\partial\varphi/\partial y = -E_y} = -E_x + iE_y = -(E_x - iE_y) = -\bar{E} \qquad (7a)$$

Alternatively: $$\frac{df}{dz} = \frac{\partial \varphi}{i\partial y} - i\frac{\partial A}{i\partial y} = -i\underbrace{\frac{\partial \varphi}{\partial y}}_{-E_y} \underbrace{-\frac{\partial A}{\partial y}}_{-\frac{\partial\varphi}{\partial x} = E_x} = -(E_x - iE_y) = -\bar{E} \qquad (7b)$$

<u>Application</u>: We show how the fact that $\varphi(x,y)$ is the real part of an analytic function can be utilized, by calculating the field of a grounded conductor formed into a right angle:



We want to find an analytic function $f(z) = \varphi(x,y) - iA(x,y)$ whose real part vanishes along $x = y = 0$.

Guess: $\varphi(x,y) = kxy \qquad k = \text{constant}$

We then guess that this is the real part of the analytic function

$$f(z) = \frac{-i}{2} k z^2 \qquad (8)$$

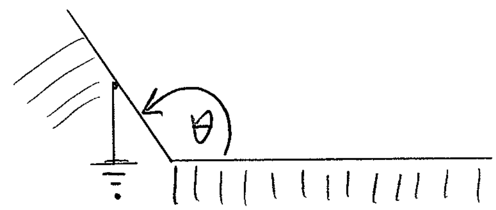
Check: $f(z) = \frac{-i}{2}k(x+iy)(x+iy) = \frac{-i}{2}k\left[(x^2-y^2) + 2ixy\right] = xy\cdot k - \frac{i}{2}k(x^2-y^2)$

$$\therefore \boxed{f(z) = \frac{-i}{2}kz^2 = \underbrace{kxy}_{\varphi} \underbrace{- \frac{i}{2}k(x^2-y^2)}_{-iA}} \qquad (9)$$

Then $$E_x = -\frac{\partial \varphi}{\partial x} = ky \quad ; \quad E_y = -\frac{\partial \varphi}{\partial y} = kx \qquad (10)$$

Conformal Transformations: (Locally angle-preserving)

Having shown that this geometry can be ~~xxxxxx~~ described by an analytic function $f(z)$, we can replace z by some function of z which has the effect of mapping this geometry into another, e.g.,



A transformation which does this is called the SCHWARZ TRANSFORMATION and an example is

$$z' = z^{\beta}$$

For appropriate choice of β this maps a flat surface into one with an angle, as shown.

For more details see PANOFSKY & PHILLIPS, Classical Electricity & Magnetism pages 66-72.

Also: E. Durand, Electrostatiques et Magnetostatiques

Elementary Analytic Functions

We describe the properties of various functions that commonly arise

a) exponential: $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$ (1)

We have shown that C-R $\Rightarrow$ e^z is analytic everywhere. We have show also that $\frac{d}{dz}e^z = e^z$ as for real functions: Note...

$$\frac{d}{dz}e^z = \frac{\partial}{\partial x}\left[e^x(\cos y + i\sin y)\right] = e^x(\cos y + i\sin y) = e^z \checkmark \quad (2a)$$

$$\text{or} \quad \frac{d}{dz}e^z = \frac{\partial}{i\partial y}\left[\quad \downarrow \quad\right] = -i\,e^x(-\sin y + i\cos y) = e^z \checkmark \quad (2b)$$

e^z is periodic with period 2π: $e^z = e^{z+2\pi i}$ (3)

b) trigonometric functions: These are defined in terms of the exponential function:

$$\boxed{\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \; ; \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)} \quad (4)$$

These are entire functions because they are expressed in terms of exponentials which are themselves entire functions. From these we have:

$$\frac{d}{dz}\sin z = \frac{1}{2i}\left(i e^{iz} + i e^{-iz}\right) = \cos z \;, \quad \text{as usual} \quad (5)$$

Other trig functions are defined as usual, but care must be taken:

$$\tan z = \frac{\sin z}{\cos z} \quad \Big\} \; \text{analytic except when } \cos z = 0 \quad (6)$$

$\therefore$ $\tan z$ is SINGULAR at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

One can verify the C-R conditions for these functions by writing

$$\sin z \equiv u(x,y) + i\,v(x,y) \quad (7)$$

$$\sin z = \frac{1}{2i}\left[e^{i(x+iy)} - e^{-i(x+iy)}\right] \qquad (8)$$

$$= \frac{1}{2i} e^{-y}(\cos x + i\sin x) - \frac{1}{2i} e^{y}(\cos x - i \sin x) \qquad (9)$$

$$= \frac{1}{2}\left(e^{y}+e^{-y}\right)\sin x + i\frac{1}{2}\left(e^{+y}-e^{-y}\right)\cos x \qquad (10)$$

$$\boxed{\therefore \sin z = \cosh y \sin x + i \sinh y \cos x = u(x,y) + i v(x,y)} \qquad (11)$$

By inspection we see that the C-R conditions hold for $\forall\ x, y$

Note also that a) $\sin iy = i \sinh y$ (12a)

b) $\overline{\sin z} = \sin \bar{z}$ (12b)

c) $\sin(z+2\pi) = \sin z$ (12c)

d) $\sin^2 z + \cos^2 z = 1$ (12d)

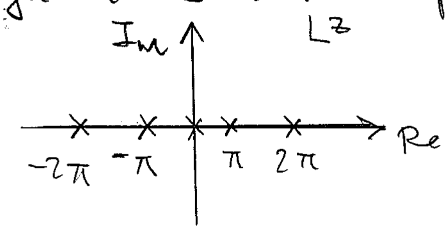
<u>Problem:</u> Find the singularities of $\csc z = \frac{1}{\sin z}$

<u>Solution:</u> Singularities occur when $\sin z = 0$

$$\sin z = \underbrace{\cosh y}_{\neq 0} \underbrace{\sin x}_{\Downarrow\ x = n\pi = 0, \pm\pi, \pm 2\pi, \dots} + i \sinh y \underbrace{\cos x}_{\text{for values where } \sin x = 0\ \cos x \neq 0}$$

$\Rightarrow \sinh y = 0 \Rightarrow y = 0$

Hence the singularities in the complex plane are at $x = \pm n\pi,\ y = 0$:



<u>Difference from the Real Case:</u> Eq. (11) $\Rightarrow |\sin z|^2 = \cancel{\cos}\,(1-\cos^2 x)(1+\sinh^2 y) + \cos^2 x \sinh^2 y$

$$= \underbrace{1-\cos^2 x}_{\sin^2 x} + \sinh^2 y - \cos^2 x \sinh^2 y + \cos^2 x \sinh^2 y$$

$$\Rightarrow \boxed{|\sin z|^2 = \sin^2 x + \sinh^2 y \Rightarrow \text{not bounded} \Big\}\ \text{general theorem!}}$$

The Complex Logarithmic Function (Intro → BRANCHES)

The multivaluedness of the angle θ begins to raise problems here.
Write $z = re^{i\theta}$ or

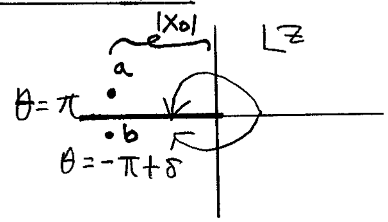
$$z = re^{i(\theta \pm 2n\pi)} \qquad n = \text{integer} \qquad (1)$$

$$\boxed{\log z = \log\left[re^{i(\theta \pm 2n\pi)}\right] \equiv \log r + i(\theta \pm 2n\pi)} \qquad (2)$$

Problem: Different values of n will lead to different numerical values for the imaginary part of $\log z$, which is therefore <u>multivalued</u>

<u>Principal Value of $\log z$</u> : take $n \equiv 0$ and $-\pi < \theta \leq \pi$

"discontinuity" of 2π across neg x-axis



← like crossing the "international date line"

↳ for the point <u>a</u> as shown: $\log z = \log|x_0| + i\pi$ (3)
for the point <u>b</u> as shown $\log z = \log|x_0| - i\pi$

However $\log z$ is defined there is a ray extending from $z=0$ to $z=\infty$ along which $\log z$ is not defined, and where it has no derivative.
Elsewhere we have

$$\log z = \log r + i\theta = \underbrace{\log\sqrt{x^2+y^2}}_{u} + i\,\underbrace{\tan^{-1}\frac{y}{x}}_{V} \qquad (4)$$

Then
$$\frac{d}{dz}\log z = \frac{\partial u}{\partial x} + i\frac{\partial V}{\partial x} = \frac{x}{x^2+y^2} + i\frac{(-y)}{x^2+y^2} = \frac{x-iy}{x^2+y^2} \qquad (5)$$
$x^2+y^2 \to z\bar{z}$

$$\therefore \boxed{\frac{d}{dz}\log z = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}} \quad (6) \qquad [\text{when } z \neq 0]$$

✓

Side Comment

$$\log z = \underbrace{\log\sqrt{x^2+y^2}}_{u(x,y)} + i\,\underbrace{\tan^{-1}\frac{y}{x}}_{v(x,y)}$$

If you forget how to differentiate $\tan^{-1} y/x$ or $\tan^{-1} x$ or $\tan^{-1} y$ just recall that $\tan^{-1}$ is the Im part of the same analytic function of which $\log\sqrt{x^2+y^2}$ is the Re part. Since it is easy to remember how to differentiate the $\log\sqrt{\cdots}$ one can use the C-R conditions to help us remember:

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} \overset{C-R}{=} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log\sqrt{x^2+y^2} = \frac{x}{x^2+y^2} \quad \text{etc.}$$

Check on Analyticity:

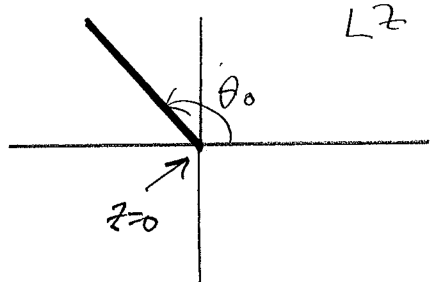
$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2} \quad ; \quad \frac{\partial v}{\partial y} = \frac{1}{(1+y^2/x^2)}\frac{1}{x} = \frac{x}{x^2+y^2} \quad \checkmark$$

Symmetry

$$\frac{\partial v}{\partial x} = \frac{1}{(1+y^2/x^2)}(-y/x^2) = \frac{-y}{x^2+y^2} \quad ; \quad -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2} \quad \checkmark$$

BRANCHES, BRANCH CUTS & BRANCH POINTS

$\log z$ can be made <u>single-valued</u> by choosing any ray (defined by θ_0) along which we restrict θ: Pictorially



$$\boxed{\theta_0 \le \theta < \theta_0 + 2\pi} \quad (1)$$

Each value of θ_0 defines a <u>branch</u> of $\log z$: A branch $F(z)$ of a multi-valued function $f(z)$ is any single-valued function in some domain where $F(z)$ coincides with $f(z)$. The choice $\theta_0 = -\pi$ defines the <u>principal branch.</u> The point $z=0$, which is common to all branches is called a <u>branch point.</u> The branch point is a <u>singular point</u> of the function $\log z$, as is every point along the ray defining the function; this ray is called a <u>branch cut</u>. At a singular point, the function is not well defined. Away from these singular points we can deal with $\log z$ as an analytic function. Thus:

$$e^{\log z} = e^{[\log r + i(\theta \pm 2n\pi)]} = \underbrace{e^{\log r}}_{r}\, e^{i\theta}\, \underbrace{e^{\pm i 2n\pi}}_{1} = re^{i\theta} = z \checkmark \quad (2)$$

Also: z^c (z, c are both complex) $\equiv [e^{\log z}]^c = e^{c \log z}$

$$\boxed{z^c \equiv e^{c \log z}} \Rightarrow (i)^i = e^{i \log i} = e^{i[\overset{0}{\log 1} + i(\overset{\pi/2}{\theta} \pm 2n\pi)]} \quad (3)$$

$$\therefore\ (i)^i = e^{-\frac{\pi}{2}}\, e^{\mp 2n\pi} \Rightarrow \boxed{(i)^i \xrightarrow{\text{Principal Value}} e^{-\pi/2}} \quad (4)$$

- Note that we are here expressing the function z^c in terms of e^z and $\log z$, so once we understand their analytic properties we can determine the analytic properties of other functions

- Note also that $$\frac{d}{dz} z^c = \frac{d}{dz}\left[e^{c\log z}\right] = e^{c\log z}\cdot\frac{c}{\underset{\Rightarrow\, e^{\log z}}{z}} \tag{5}$$

$$= c\,\frac{e^{c\log z}}{e^{\log z}} = c\,e^{(c-1)\log z} = c\,z^{c-1}\ \checkmark \tag{6}$$

Analysis of $f(z) = z^{1/2}$; using $e^{c\log z} = z^c$ we have:

$$z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\ln r + i\theta)} \Rightarrow \underbrace{e^{\frac{1}{2}\ln r}}_{r^{1/2}}\, e^{i\theta/2}\, e^{i\frac{1}{2}(\pm 2n\pi)} \tag{7}$$

$$\boxed{z^{1/2} = r^{1/2}\, e^{i\theta/2}\, e^{\pm i\pi n}} \tag{8}$$

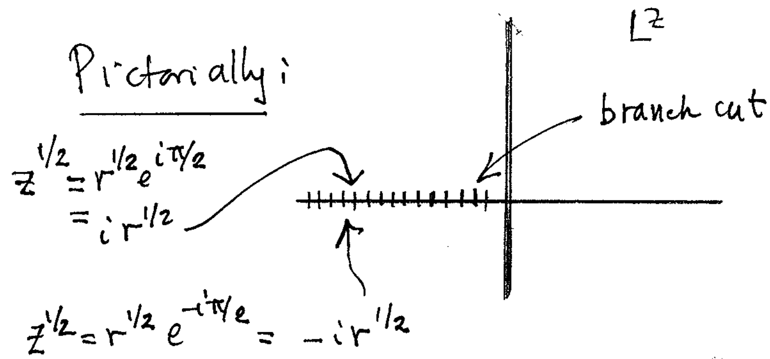
$$\boxed{\therefore\ z^{1/2} = r^{1/2}\, e^{i\theta/2}\,(-1)^n = \pm\, r^{1/2}\, e^{i\theta/2}} \tag{9}$$

So $z^{1/2}$ is double-valued just like $\sqrt{4} = \pm 2$ is double-valued. For an arbitrary z we then have 2 branches:

Principal branch: $z^{1/2} = r^{1/2}\, e^{i(\frac{\theta}{2} \pm n\pi)} \equiv f_1 \qquad -\pi < \theta \le \pi$

Other branch: $z^{1/2} = r^{1/2}\, e^{i(\frac{\theta}{2} \pm n\pi)} \equiv f_2 \qquad -\pi \le \theta \le \pi$

Pictorially:



Discontinuity (in phase) $= \pi/2 - (-\pi/2) = \pi\ \checkmark$