INVERSE TRIGONOMETRIC FUNCTIONS:

Consider $W = \sin^{-1} z$; To study the analytic properties of $\sin^{-1} z$ we [redacted] seek to express W in terms of $\log z, \ldots$ whose analytic properties we know. To do this write:

$$W = \sin^{-1} z \Rightarrow \sin W = z = \frac{e^{iW} - e^{-iW}}{2i} \Rightarrow e^{iW} - e^{-iW} = 2iz \qquad (1)$$

mult by e^{iW}

$$\therefore e^{2iW} - 2ize^{iW} - 1 = 0 \Rightarrow (e^{iW})^2 - 2iz(e^{iW}) - 1 = 0 \qquad (2)$$

$$\xi^2 - 2iz\xi - 1 = 0$$

$$\therefore \underset{e^{iW}}{\xi} = \frac{2iz \pm (4 - 4z^2)^{1/2}}{2} = iz \pm (1 - z^2)^{1/2} \qquad (3)$$

choose + root

$$\text{Hence} \quad e^{iW} = iz \pm (1-z^2)^{1/2} \Rightarrow \boxed{W = W(z) = -i \log\left[iz \pm (1-z^2)^{1/2}\right]} \qquad (4)$$

$$\boxed{= \sin^{-1} z}$$

In this way we replace an "unknown" function of a simple argument with a known function of a more complicated argument. We can then use this relation to differentiate $W(z)$:

$$\frac{d}{dz} W(z) = \frac{d}{dz} \sin^{-1} z = \frac{-i}{iz \pm (1-z^2)^{1/2}} \left\{ i \pm \frac{1}{2}(1-z^2)^{-1/2}(-2z) \right\}$$

$$\text{choosing + root} \Rightarrow \{\ldots\} \Rightarrow \boxed{\frac{d}{dz} = \frac{d}{dz} \sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}}$$

In an analogous way one can show that

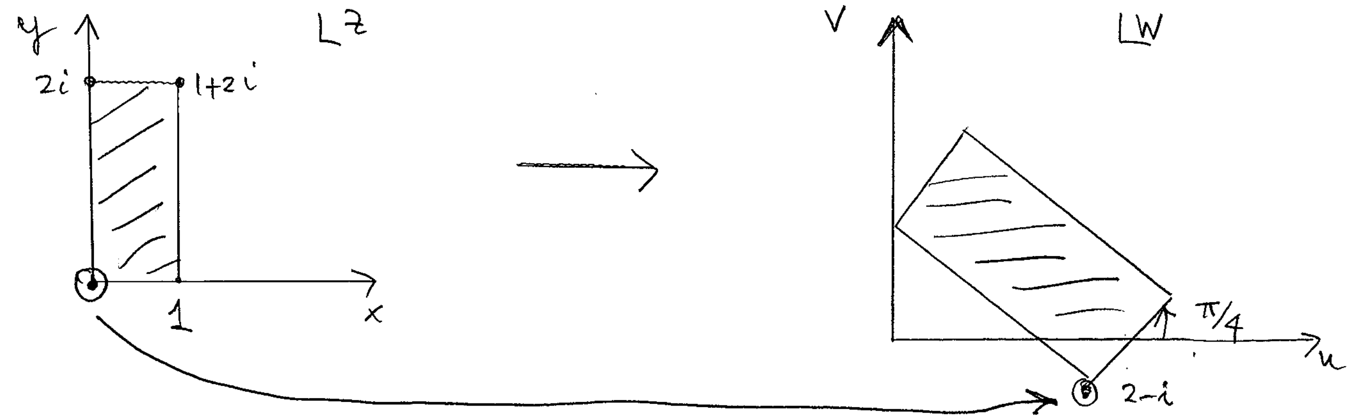
$$\boxed{\tan^{-1} z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)}$$

AN EXHIBITION OF PICTURES: GRAPHING (PICTURING) COMPLEX FUNCTIONS

To develop a physical picture of what $W = f(z)$ does, we can focus on a mapping of a portion of the LZ plane

a) $W = f(z) = Bz + c$ ← translates
↑ rotates & multiplies

Example: $w = u + iv = (1+i)z + (2-i)$



Note that $\arg(1+i) = \pi/4$:

$$\hookrightarrow re^{i\theta} \Rightarrow \theta = \arg(1+i) = \pi/4 \qquad r = |1+i| = \sqrt{2}$$

Summary: The rectangle shown in the complex z plane is first translated so that the origin → $2-i$; then it is rotated by $\pi/4$, and the lengths of the sides are multiplied by $\sqrt{2}$. All this follows immediately by writing

$$(1+i) = |1+i|\, e^{i\tan^{-1}1} = \sqrt{2}\, e^{i\pi/4}$$

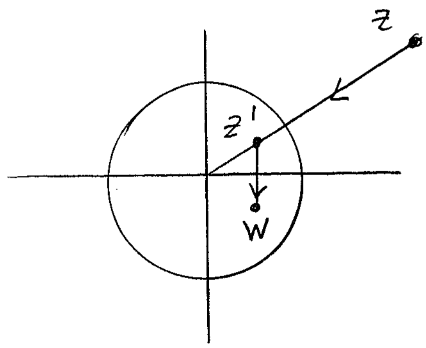
b) $W = f(z) = 1/z \equiv \rho e^{i\varphi}$ (in W-plane) (1)

$z = re^{i\theta} \Rightarrow \frac{1}{z} = \frac{1}{r}e^{-i\theta}$ (2)

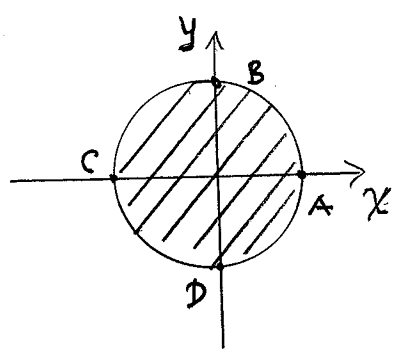
Hence (1) & (2) $\Rightarrow$ $\boxed{\rho e^{i\varphi} = \frac{1}{r}e^{-i\theta}}$ (3)

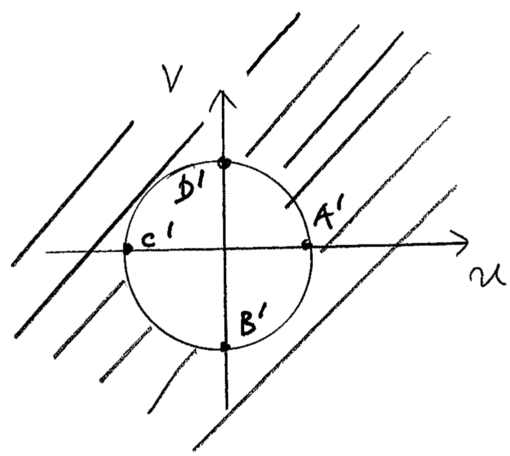
This can be pictured as the net effect of 2 successive transformations:

$$\underbrace{z' = \frac{1}{r}e^{i\theta}}_{\text{inversion with respect to unit circle}} \rightarrow W = \bar{z}' = \frac{1}{r}e^{-i\theta}$$



This maps the inside of the circle to the outside & vice versa (see below)





Note that points along the unit circle map into the unit circle except that they are inverted relative to the x-axis:

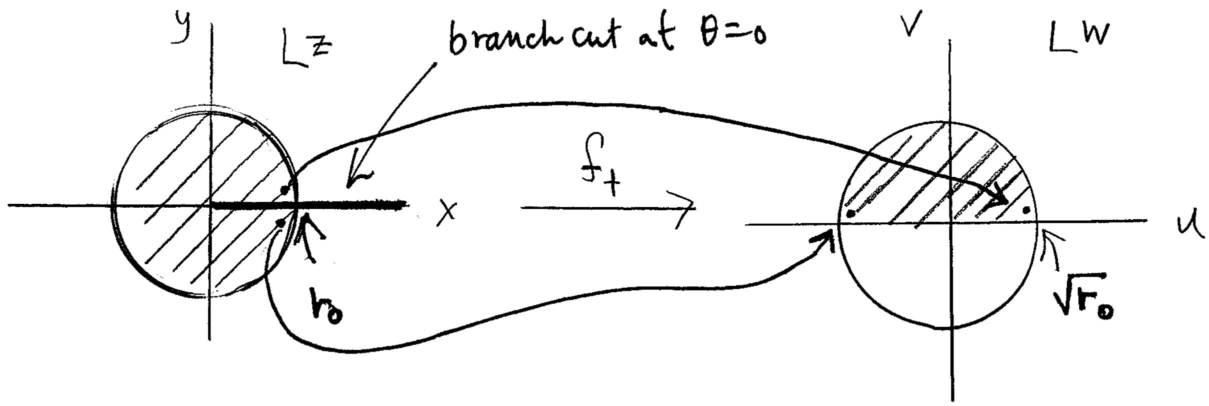
upper half plane $\rightarrow$ lower half plane.

Note also that $W = 0 \Leftrightarrow z = \infty$. This means that the transformation $W = 1/z$ is useful in studying the $z \to \infty$ limit of a complex function.

Example: $W = \dfrac{4z^2}{(1-z)^2} \Rightarrow$ $z = \infty \Leftrightarrow W = 4$

$z = 1 \Leftrightarrow W = \infty$

c) $f(z) = w = z^{1/2}$



As noted previously, there are 2 branches $f_\pm$ to this function

$$f_\pm = \pm\sqrt{r}\, e^{i\theta/2} \qquad (0 \leq \theta < 2\pi)$$

Hence a point on or just above the real axis maps to a point in a similar location as shown. However, a point just below the real axis in the LZ plane maps to a point along the negative real axis as shown. This is again a reflection of the discontinuity that arises from the presence of the branch cut.

Note that wherever the branch cut is taken to be, there will be some similar discontinuity.

COMPLEX INTEGRATION

We wish to consider integrals of the form

$$\int_C W(z)\,dz = \int [u(x,y) + iV(x,y)][dx + idy] \qquad (1)$$

$z \rightarrow C$

Some contour or path

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

If x and y are literally 2-dim coordinates, they may depend parametrically on another coordinate t (e.g. time) in which case we have

$$\int_C W(z)\,dz = \int_{t_0}^{t_1} dt\left(u\frac{\partial x}{\partial t} - v\frac{\partial y}{\partial t}\right) + i\int_{t_0}^{t_1} dt\left(v\frac{\partial x}{\partial t} + u\frac{\partial y}{\partial t}\right) \qquad (2)$$

We define $\int_c = -\int_{-c}$ ← contour traversed in opposite direction

<u>Definition</u>: Unless otherwise stated $\int_C$ for a closed contour is taken to be in the <u>counter clockwise</u> (CCW) direction.

Also: $\int_{C_1} + \int_{C_2} = \int_{C_1+C_2}$ (obvious!)

<u>Triangle Inequality for Integrals:</u>

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz| \qquad (3)$$

This can be seen as follows:

$$\int f(z)\,dz \cong \sum_i \{f(z_1)\Delta z + f(z_2)\Delta z + \dots\} \qquad (4)$$

$$= \sum \{f(z_1) + f(z_2) + \dots\}\Delta z \Rightarrow \left|\sum \{f(z_1) + f(z_2) + \dots\}\Delta z\right| \leq \sum_i |f(z_i)||\Delta z| \qquad (5)$$

cancellations possible

no cancellations possible

Hence $\left|\int_C f(z)\,dz\right| \leq \int |f(z)||dz|$

Recall that the usual triangle inequality is:

$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad (6)$$

Note that (5) can also be arrived at by writing

$$\int_C \cong \sum \{f(z_1)\Delta z + f(z_2)\Delta z + \dots\} \Rightarrow \qquad (7)$$

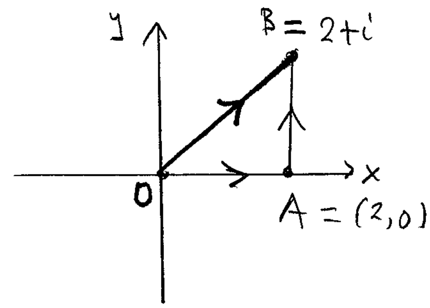
$$\left|\sum\{\dots\}\right| = |f(z_1)\Delta z + f(z_2)\Delta z + \dots| \leq |f(z_1)\Delta z| + |f(z_2)\Delta z| + \dots$$

$$\text{But } |f(z_1)\Delta z| = |f(z_1)||\Delta z| \Rightarrow \sum_i \leq \sum_i |f(z_i)||\Delta z| \qquad (8)$$

We will return to use this later;

<u>Example of Complex Integration:</u>

Find $I = \int_C z^2 dz$



C is <u>either</u> 1) OB $\equiv C_1$

2) OA + AB $\equiv C_2$

<u>NOTE:</u> When a path or contour is specified in the $\angle z$ plane this defines a relationship between x and y along the path (or between r and θ in polar coordinates) so that there is <u>only 1 independent integration variable left.</u>

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1) Along C_1 : $z^2 = \underbrace{x^2-y^2}_{u} + i\underbrace{2xy}_{v}$; $dz = dx + i\,dy$

$$\int_{C_1} z^2\,dz = \int_C \left[(x^2-y^2)\,dx - 2xy\,dy\right] + i\int_{C_1} (x^2-y^2)\,dy + 2xy\,dx$$

Up to this point no detailed specification of the path has taken place (i.e. no relation between x & y).

Next we note that along $C_1 = OB$ $x = 2y \Rightarrow dx = 2dy$

Hence $$\int_{C_1} = \int_{y=0}^{y=1} \left(3y^2 \cdot 2dy - 4y^2 \cdot dy + i\,3y^2 \cdot dy + i\,4y^2 \cdot 2dy\right) = \frac{2}{3} + i\frac{11}{3}$$

2) $$\int_{C_2} = \int_{OA} z^2\,dz + \int_{AB} z^2\,dz$$

Along OA $\left\{\begin{matrix} dy = 0 \\ y = 0 \end{matrix}\right\} \Rightarrow \int_{OA} z^2\,dz \to \int_0^2 u\,dx + i\int_0^2 v\,dx$

Since $y = 0 \Rightarrow v = 0$ $= \int_{OA} = \int_0^2 x^2\,dx + i\int 0 = \frac{8}{3}$

Along AB $\left\{\begin{matrix} dx = 0 \\ x = 2 \end{matrix}\right\} = \int_{AB} = \int_0^1 -v\,dy + i\int u\,dy = -4\int_0^1 y\,dy + i\int_0^1 (4-y^2)\,dy$

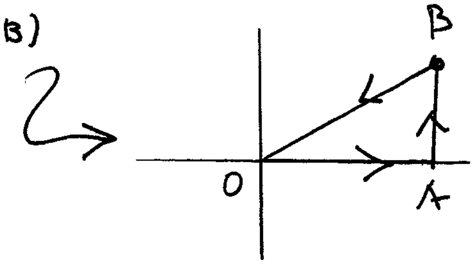
$= -2 + i\frac{11}{3}$

Hence $$\int_{C_2} = \int_{OA} + \int_{AB} = \frac{8}{3} + \left(-2 + i\frac{11}{3}\right) = \frac{2}{3} + i\frac{11}{3}$$

This is an example of a theorem we are about to prove: $\int_{z_1}^{z_2} f(z)\,dz$ is <u>independent of the path</u> <u>if $f(z)$ is analytic.</u> Note also that for the closed path $OA+AB+(-OB)$

We have:

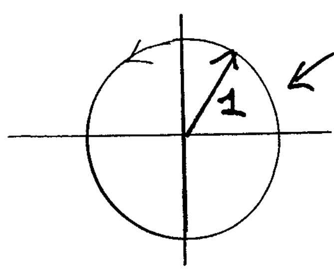
$$\oint_{OA+AB+(-OB)} = \left(\frac{2}{3}+i\frac{11}{3}\right)-\left(\frac{2}{3}+i\frac{11}{3}\right)=0$$



Similarly: <u>for $f(z)$ analytic</u>

$$\boxed{\oint f(z)\,dz = 0}$$

<u>Another Example:</u> Evaluate $\oint_C \bar{z}\,dz$ $\quad C =$ Unit circle around origin



$$\Rightarrow \oint_C \bar{z}\,dz = \int_0^{2\pi} \underbrace{1e^{-i\theta}}_{\bar{z}} \cdot \underbrace{(ie^{i\theta}d\theta)}_{dz \text{ along contour}}$$

$$= 1\int_0^{2\pi} i\,d\theta = 2\pi i \neq 0$$

Since $\bar{z}$ is not analytic $\oint \neq 0$, in contrast to previous example.

Note also: Along unit circle $\bar{z} = 1e^{-i\theta} = 1/z$. Hence (for later) we have also shown that

$$\oint_C \frac{1}{z}\,dz = 2\pi i$$

Side Comment: Since $f(z)=z^2$ is analytic one can also evaluate the integral directly as in real integration:

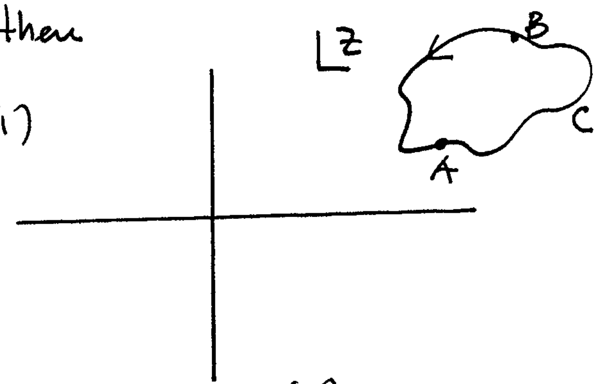
$$I=\int_0^{2+i} dz\, z^2 = \frac{1}{3}z^3\Big|_0^{2+i} = \frac{1}{3}(2+i)^3 = \frac{1}{3}(2+11i) = \frac{2}{3}+i\frac{11}{3} \checkmark$$

This is a fundamental theorem of complex calculus which holds in any regime where $f(z)$ is analytic.

CAUCHY'S THEOREM: (IMPORTANT!!)

Thm: If $f(z)$ is analytic within and on a contour C (closed) and $f'(z)$ is continuous in this region then

$$\oint_C f(z)\,dz = 0 \qquad (1)$$



Proof: $$\oint f(z)\,dz = \oint (u\,dx - v\,dy) + i\oint (u\,dy + v\,dx) \qquad (2)$$

We can show that $\oint = 0$ if we can show that the integrand is a perfect differential; for example:

$$\oint (u\,dx - v\,dy) = \oint d\phi = \int_A^B d\phi + \int_B^A d\phi = (\phi_B - \phi_A) + (\phi_A - \phi_B) = 0 \qquad (3)$$

To show this, consider a scalar function $\phi(x,y)$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy \equiv \Phi_x\,dx + \Phi_y\,dy \qquad (4)$$

Assuming all relevant derivatives exist, then

$$\frac{\partial \Phi_x}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \qquad \frac{\partial \Phi_y}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} \Rightarrow \boxed{\frac{\partial \Phi_x}{\partial y} = \frac{\partial \Phi_y}{\partial x}} \qquad (5)$$

Hence from (4) & (5):

$$\boxed{\Phi_x\,dx + \Phi_y\,dy = d\phi \Rightarrow \frac{\partial \Phi_x}{\partial y} = \frac{\partial \Phi_y}{\partial x}} \qquad (6)$$

We can also show that the implication goes the other way: Let $\vec{\Phi} = (\Phi_x, \Phi_y, 0)$

We have shown at the beginning of the semester that

$$\vec{\nabla}\times\vec{\Phi} = 0 \Leftrightarrow \vec{\Phi} = \vec{\nabla}\phi \qquad (7)$$

Now if $\dfrac{\partial \Phi_x}{\partial y} = \dfrac{\partial \Phi_y}{\partial x} \Rightarrow \partial_x \Phi_y - \partial_y \Phi_x = (\vec{\nabla}\times\vec{\Phi})_z = 0 \qquad (8)$