Furthermore, in 2-dimensions where $\Phi = \Phi(x,y)$ we have

$$(\vec{\nabla}\times\vec{\Phi})_x = \underset{0}{\underset{\parallel}{\partial_y \Phi_z}} - \underset{0}{\underset{\parallel}{\partial_z \Phi_y}} \quad ; \quad (\vec{\nabla}\times\Phi)_y = \underset{0}{\underset{\parallel}{\partial_z \Phi_x}} - \underset{0}{\underset{\parallel}{\partial_x \Phi_z}} \quad (9)$$

Hence altogether: $(\vec{\nabla}\times\Phi)_x = (\vec{\nabla}\times\vec{\Phi})_y = (\vec{\nabla}\times\vec{\Phi})_z = 0 \Rightarrow (\vec{\nabla}\times\vec{\Phi}) = 0$

$$\Downarrow$$

$$\Phi = \vec{\nabla}\phi \quad (10)$$

Combining the previous results Cauchy's Theorem follows by the following chain of arguments:

$$\partial_y \Phi_x(x,y) \overset{①}{=} \partial_x \Phi_y(x,y) \Rightarrow \overset{②}{\vec{\nabla}\times\vec{\Phi} = 0} \Rightarrow \overset{③}{\vec{\Phi} = \vec{\nabla}\phi} \Rightarrow \quad (11)$$

$$d\phi \overset{④}{=} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \overset{⑤}{\equiv} \Phi_x\, dx + \Phi_y\, dy \quad (12)$$

$$\therefore \quad \Phi_x dx + \Phi_y dy \overset{⑥}{=} d\phi = \underline{\text{perfect}}\ \underline{\text{differential}} \quad (13)$$

<u>Stated In Words:</u>

a) If $\partial\Phi_x/\partial y = \partial\Phi_y/\partial x$ this implies $\vec{\nabla}\times\vec{\Phi} = 0$

b) If $\vec{\nabla}\times\vec{\Phi} = 0$ then $\vec{\Phi}$ can be written as $\Phi = \vec{\nabla}\phi$

(this introduces the scalar field ϕ)

c) Since ϕ is a scalar, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \equiv \Phi_x dx + \Phi_y dy$

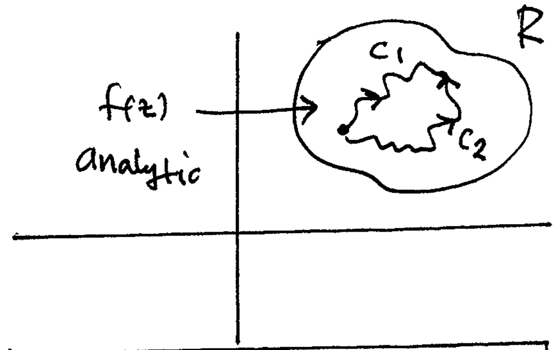
d) Combining a) ... c) we see that an expression such as $(\Phi_x dx + \Phi_y dy)$ is a perfect differential if $(\partial_y \Phi_x = \partial_x \Phi_y)$

e) If we identify $u(x,y) = \Phi_y(x,y)$ and $v(x,y) = \Phi_x(x,y)$ then the condition for a perfect differential is just the C-R ~~[illegible]~~ equation $\partial u/\partial x = \partial v/\partial y$. The same holds true for the other C-R relation

f) From Eq. (3) p. CV-34.1 this completes the proof.

I IMPLICATIONS OF CAUCHY THEOREM

a) Path Independence of $\int f(z)dz$



Since $\oint_{c_1+(-c_2)} f(z)\,dz = 0$

$$\Rightarrow \int_{c_1} f(z)dz + \int_{-c_2} f(z)dz = 0 \Rightarrow \boxed{\int_{c_1} f(z)dz = \int_{c_2} f(z)dz} \quad (1)$$

This is the same statement as the path-independence of the work done moving in a "conservative-field" (e.g. gravity). The reasons are also the same (... perfect differential...)

b) Fundamental Theorem of Calculus (*)

From a) above the function $F(z) \equiv \int_{z_0}^{z} f(z')dz'$ defines a unique function, since all that needs to be specified are the endpoints z_0, z

(*) **Theorem**: $F(z)$ is also analytic and $F'(z) = f(z)$ (2)

Proof: $F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} \cdots - \int_{z_0}^{z} \cdots = \int_{z}^{z+\Delta z} \cdots$ along any path (e.g. a line)

Note that trivially: $$f(z) = \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} dz' = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,dz' \quad (3)$$ (no prime!)

Then (2) & (3) $\Rightarrow$ $$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z') - f(z)]\,dz' \quad (4)$$

we want to show that this $\to 0$

We note that $f(z)$ is continuous (because it is analytic) and hence for all $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(z') - f(z)| < \epsilon \quad \text{when} \quad |z' - z| < \delta \tag{5}$$

In our case $z' - z = \Delta z$, so take $0 < |\Delta z| < \delta$

$$\text{Then} \quad \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \int_z^{z+\Delta z} |\underbrace{f(z') - f(z)}_{<\epsilon}| \, |dz'| \tag{6}$$

[The above uses the triangle inequality that we previously proved for integrals]

$$\text{From (6):} \quad \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{\epsilon}{|\Delta z|} \underbrace{\int_z^{z+\Delta z} |dz'|}_{\Delta z} = \epsilon \tag{7}$$

$$\text{Hence} \quad \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \epsilon \quad \text{when} \quad 0 < |\Delta z| < \delta \tag{8}$$

Now as $|\Delta z| \to 0$ $\quad \delta \to 0 \Rightarrow \epsilon \to 0$ so that (9)

$$\left| \underbrace{\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}}_{F'(z)} - f(z) \right| \to 0 \quad \Rightarrow \boxed{F'(z) = f(z)}$$

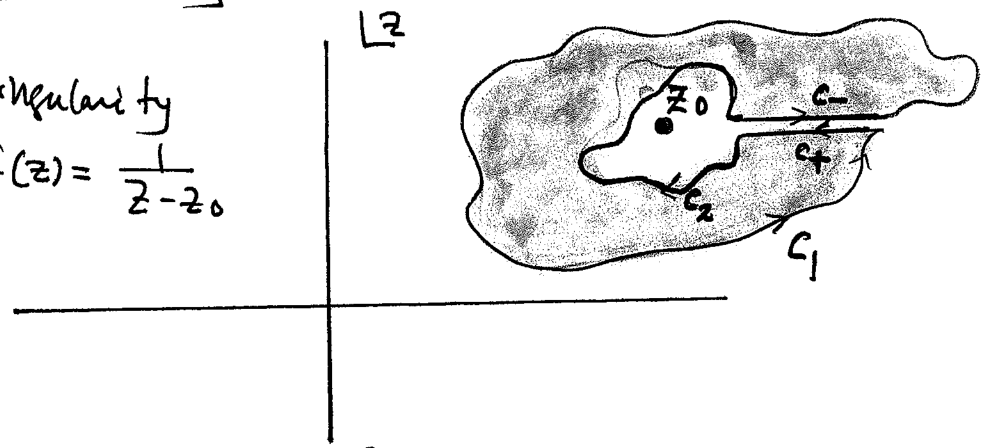
The usual version of the fundamental theorem of calculus then follows:

$$\int_\alpha^\beta f(z)\,dz = \int_{z_0}^{\beta} f(z)\,dz' - \int_{z_0}^{\alpha} f(z)\,dz = F(\beta) - F(\alpha) \quad \text{where} \tag{10}$$

$$F(z) = \int_{z_0}^{z} f(z')\,dz' \tag{11}$$

(c) Contours Containing Singularities:

$f(z)$ has a singularity at z_0, e.g. $f(z) = \frac{1}{z - z_0}$



If the task is to evaluate $\oint_{C_1} f(z)\,dz$, where C_1 is some complicated contour, we can deform the contour as shown: The contributions along C_+ and C_- cancel (since there $f(z)$ is analytic).

Moreover $\displaystyle\int_{C_1 + C_+ + C_- + (-C_2)} = 0$ } Since there are no singularities inside this contour.

Since $\int_{C_+} + \int_{C_-} \equiv 0 \Rightarrow \int_{C_1} + \int_{-C_2} \equiv 0 \Rightarrow \boxed{\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz}$

Note that this is a different result from that proved in a) on p.39, since there $f(z)$ was analytic in the region R, but here it is not: This is what is called a "multiply connected region" where the contour encloses a region within which $f(z)$ is not analytic, and where $\oint_{C_1} f(z)\,dz \neq 0$.

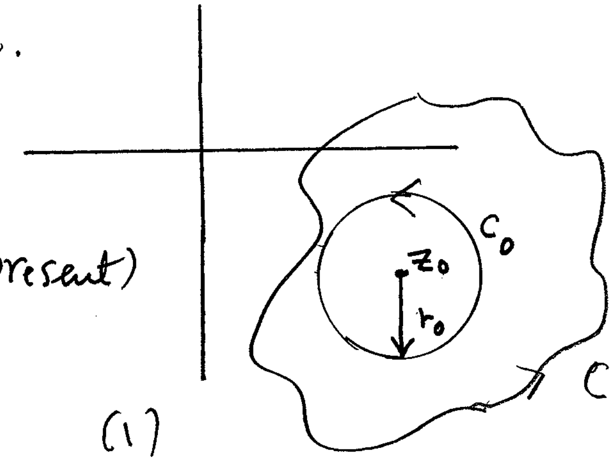
Implications: Given C_1 you can make your life much easier by replacing C_1 by a simpler contour C_2 (like a circle), provided that C_1 and C_2 enclose exactly the same singularities.

d) CAUCHY'S INTEGRAL FORMULA *

* VERY IMPORTANT!!

Let $g(z)$ denote a general function of z, which may or may not be analytic in some domain.

Then we have shown that if C and C_0 enclose the same singularities (if any are present) then



$$\oint_C g(z)\,dz = \int_{C_0} g(z)\,dz \qquad (1)$$

C_0 ← circle of radius r_0. $\Rightarrow$ along C_0 $\boxed{z - z_0 = r_0 e^{i\theta}}$ $\boxed{dz = i r_0 e^{i\theta} d\theta}$ ☆ (2)

Of special interest are functions $g(z)$ having the form

$$\boxed{g(z) = \frac{f(z)}{z - z_0} \quad ; \ f(z) \text{ is analytic within } C} \quad (3)$$

$g(z)$ is not analytic at z_0, but has the special form of non-analyticity given in (2). To evaluate $\oint_C g(z)\,dz$ we replace C by C_0 as shown. Then:

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{C_0} dz \frac{f(z)}{z - z_0} = \underbrace{f(z_0) \oint_{C_0} \frac{dz}{z - z_0}}_{\mathrm{I}} + \underbrace{\oint_{C_0} dz \frac{[f(z) - f(z_0)]}{z - z_0}}_{\mathrm{II}} \qquad (4)$$

II: As $r_0 \to 0$ $(z - z_0) \equiv \Delta z \to 0$ and $[f(z) - f(z_0)]/\Delta z \to f'(z)$, which is analytic since $f(z)$ is analytic. Hence via Cauchy

$$\mathrm{II} = \oint dz\, f'(z) \equiv 0$$

I Hence: $\oint \frac{f(z)}{z - z_0}\,dz = f(z_0) \oint \frac{dz}{(z - z_0)} \overset{☆}{=} f(z_0) \int_0^{2\pi} d\theta \frac{(i \cancel{r_0 e^{i\theta}} d\theta)}{\cancel{r_0 e^{i\theta}}} = 2\pi i f(z_0)$

$$\therefore \quad \boxed{\oint dz \frac{f(z)}{z - z_0} = 2\pi i f(z_0)}$$

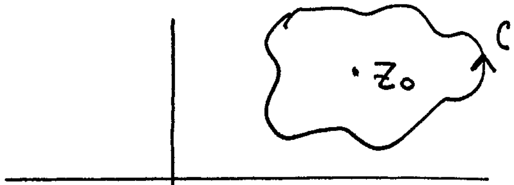
CAUCHY'S INTEGRAL FORMULA

COMMENTS:

This is one of the most important results in the theory of complex functions from both a practical & "philosophical" point of view:

<u>practical</u> : Evaluation of real integrals via contour integration

<u>"philosophical"</u>:



$$\frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{z-z_0} = f(z_0)$$

This tells us that the value of a function $f(z)$, which is analytic, can be determined at any point interior to C by knowing its value only along the boundary C of that region. [Since there are "more points" in the interior than on its boundary, analyticity buys us something.]

CONSEQUENCES OF CAUCHY'S INTEGRAL FORMULA

a) Derivatives of Analytic Functions

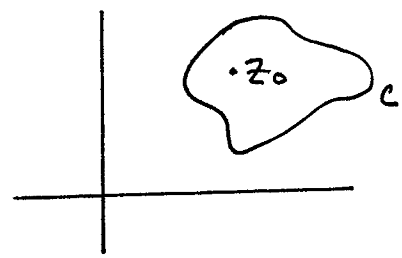
One can prove that all of the derivatives of an analytic function are analytic. This is not true for real variables: $x^{1/2}$ is differentiable everywhere, but its derivative ~ $x^{-1/2}$ has a singularity at the origin.

By contrast $z^{1/2}$ is analytic, but only because we introduced a branch cut, (e.g. along the real axis), and this eliminates the point $z=0$ where the derivative of $z^{1/2}$ would have a singularity.

To differentiate an analytic function start with CAUCHY'S INTEGRAL FORMULA:

$$\frac{d}{dz_0} f(z_0) = \frac{d}{dz_0}\left\{\frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{z-z_0}\right\} = \frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^2} \qquad (1)$$

any point inside C (a variable for these purposes)



Similarly, taking another derivative:

$$\frac{d^2}{dz_0^2} f(z_0) \equiv f''(z_0) = \frac{2!}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^3} \qquad (2)$$

For the nth derivative:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^{n+1}}} \qquad (3)$$

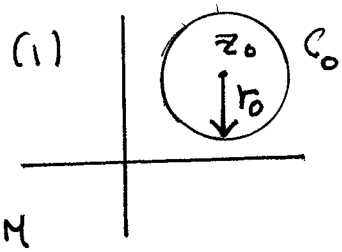
Thus $f(z)$ has all possible derivatives within C. The kth derivative is therefore continuous in C because this formula allows the $(k+1)$th derivative to be computed.

b) LIOUVILLE'S THEOREM:

If $f(z)$ is analytic and $|f(z)|$ is bounded for all values of z, then $f(z) = \text{constant}$.

Proof: Start with CAUCHY'S INTEGRAL FORMULA ⇒

$$f'(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{(z-z_0)^2} \qquad (1)$$



Choose this contour for $C = C_0$

Then: $|z - z_0| = r_0$ and

$$|f'(z_0)| \leq \left|\frac{1}{2\pi i}\right| \oint_{C_0} \underbrace{|dz|}_{2\pi r_0} \frac{\overbrace{|f(z)|}^{\leq M}}{\underbrace{|(z-z_0)|^2}_{r_0^2}} \leq \frac{1}{2\pi r_0^2} \cdot M \cdot 2\pi r_0 = \frac{M}{r_0} \qquad (2)$$

Here M denotes the maximum value that $f(z)$ assumes in the complex plane, (which we can do since by assumption $|f(z)| \leq M$).

From Eq. (2) above it follows that

$$|f'(z_0)| \leq \frac{M}{r_0} \qquad \leftarrow \text{for any } r_0$$

∴ Take $r_0 \to \infty \Rightarrow |f'(z_0)| \to 0 \Rightarrow f(z_0) = \text{constant}$

Q.E.D.

Implication: If an analytic function is not a constant, then it cannot be bounded. Recall the example given previously [p. CV-20,21]

$$\sin z = \sin x \cdot \cosh y + i \sinh y \cos x$$

$$|\sin z|^2 = \sin^2 x + \underbrace{\sinh^2 y}_{\text{not bounded}}$$

(C) FUNDAMENTAL THEOREM OF ALGEBRA

If $P_m(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$ is a polynomial in z

Then $P_m(z) = 0$ has at least one root.

Proof: Assume the contrary – that $P_m(z) \neq 0$ for any z (i.e. that there is no root). Then $P_m(z)$ is entire (i.e. analytic in the entire complex plane.) Moreover, since $P_m(z)$ is a polynomial, $|1/P_m(z)| \to 0$ as $z \to \infty \Rightarrow P_m(z)$ is bounded for all z. This follows by noting that in the finite part of the $\angle z$ plane we can find the biggest value of z and be assured that there will not be a larger value as $|z| \to \infty$.

It follows that since $|1/P_m(z)|$ is bounded, then if we assume $P_m(z) \neq 0$ anywhere, so that $1/P_m(z)$ is analytic everywhere, then $1/P_m(z)$ must be a constant $\Rightarrow$ Contradiction!

It then follows that the assumption that $P_m(z)$ does not vanish anywhere must be false $\Rightarrow$ for some z_0 $P_m(z_0) = 0$. QED ✓

This argument can be repeated by writing

$$P_m(z) = (z - z_0) P_{m-1}(z)$$

$P_{m-1}(z)$ must then have a root also, at $z_1 \Rightarrow P_m(z) = (z - z_0)(z - z_1) P_{m-2}(z)$.

Hence altogether:

$$\boxed{P_m(z) = \underbrace{(z - z_0)(z - z_1) \cdots (z - z_{m-1})}_{m \text{ factors}}}$$