The K-K dispersion relations were originally derived for the electric susceptibility $\chi(z) \to \chi_R(\omega) + i\chi_I(\omega)$. However, they are very widely applied in many areas including Condensed matter and high-energy physics.

Subtracted Dispersion Relations:

The Kramers-Kronig dispersion relations would appear to be useless in cases where $u(\omega')$ or $v(\omega')$ does not vanish sufficiently fast at ∞. In this case the integrals in (10) and (14) might not converge. Even if the integrals formally converge, we may want them to converge faster for computational purposes. Subtracted dispersion relations is a technique for increasing the rate of convergence:

Suppose we know the value of $u(\omega)$ at some $\omega = \omega_0$:

$$u(\omega_0) = \frac{2}{\pi} P \int_0^\infty d\omega' \, \frac{\omega' v(\omega')}{\omega'^2 - \omega_0^2} \tag{15}$$

Then

$$u(\omega) = \underbrace{u(\omega_0) - \frac{2}{\pi} P \int_0 d\omega' \, \frac{\omega' v(\omega')}{\omega'^2 - \omega_0^2}}_{=0} + \frac{2}{\pi} P \int_0 d\omega' \, \frac{\omega' v(\omega')}{\omega'^2 - \omega^2} \tag{16}$$

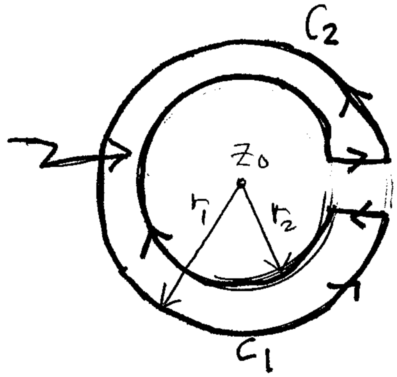
$$u(\omega) = u(\omega_0) + \frac{2}{\pi} P \int_0^\infty d\omega' \, \omega' v(\omega') \left\{ \frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \omega_0^2} \right\} \tag{17}$$

$$\{\ldots\} = \frac{\omega^2 - \omega_0^2}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)} \Rightarrow \boxed{u(\omega) = u(\omega_0) + \frac{2}{\pi}(\omega^2 - \omega_0^2) \int_0^\infty d\omega' \, \frac{\omega' v(\omega')}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)}} \tag{18}$$

Convergence of $\int_0^\infty d\omega'$ is improved because the denominator now goes as $(1/\omega')^4$ for large ω', rather than $(1/\omega')^2$; This process can be repeated.

TAYLOR & LAURENT SERIES

f(z) is analytic here



LAURENT'S THEOREM (GENERALIZATION OF TAYLOR EXPANSION)

If $f(z)$ is analytic in the region between C_1 and C_2 then $f(z)$ can be expanded in this region as:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n \;;\; A_n = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}} \quad (1)$$

note presence of negative powers

any closed contour circling z_0

Comments:

(1) The proof is direct, but will not be given in class. [See BYRON/FULLER p. 349]

(2) The Laurent theorem allows for the presence of singularities by including negative powers of $(z'-z_0)$.

(3) For a function $f(z)$ which is analytic everywhere we have:

(a) for $n \leq -1$

$$A_n = \frac{1}{2\pi i}\oint_C dz'\, f(z')\,(z'-z_0)^{-n-1} \quad (2)$$

$$= \frac{1}{2\pi i}\oint_C dz'\, \underbrace{f(z')\,\underbrace{(z'-z_0)^{|n|-1}}_{1 \text{ or positive power of } (z'-z_0)}}_{\text{analytic}} \equiv 0 \quad (3)$$

(b) For $n \geq 0$ we have

$$A_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0) \quad \Bigg\} \text{ CAUCHY INTEGRAL FORMULA} \quad (4)$$

Hence altogether, if $f(z)$ is analytic

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z-z_0)^n} \quad (5)$$

TAYLOR'S THEOREM

(4) When a singular function can be expanded by inspection into various inverse powers, then this must be the result obtained by formally evaluating $\oint_C$, since an expansion about a singular point is unique. Example: [B/F p.352]

$$f(z) = \frac{z^3+2z^2+4}{(z-1)^3} = 1 + \frac{5}{z-1} + \frac{7}{(z-1)^2} + \frac{7}{(z-1)^3} \quad (6)$$

Examples of a Laurent Expansion:

$f(z) = \cosh\left(z + \frac{1}{z}\right)$; this is analytic everywhere except at $z=0$.

Choose C_2 = arbitrarily small circle about origin, and C_1 to be arbitrarily large. Then choose C = unit circle about the origin:

$$z' - z_0 = 1\, e^{i\theta} \qquad dz' = i\, e^{i\theta} d\theta \quad (7)$$

$$z' + \frac{1}{z'} = e^{i\theta} + \frac{1}{e^{i\theta}} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

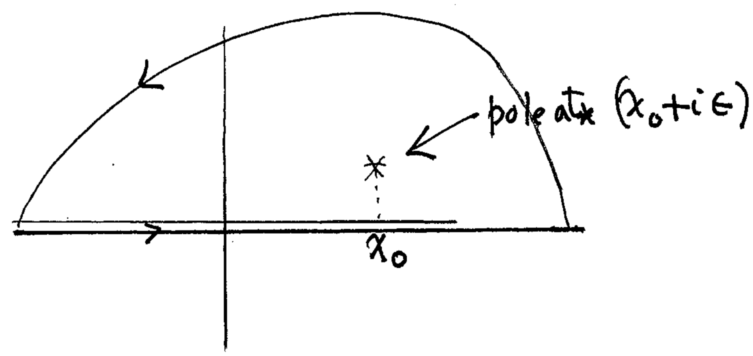
$$\frac{1}{z'^{n+1}} \equiv \frac{1}{(z'-z_0)^{n+1}} = e^{-in\theta}\, e^{-i\theta} \quad (8)$$

Collecting together the results in (5) & (6) we have:

$$\boxed{I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0} = 2\pi i f(x_0) - \pi i f(x_0) = \pi i\, f(x_0)} \qquad (7)$$

We can infer from this a "rule":

A pole of the form $\frac{f(z)}{z-z_0}$ in the complex plane (but not on the real axis) contributes a reside $2\pi i f(z_0)$ at the pole. <u>However, if the pole lies on the real axis its contribution is $\pi i f(x_0)$.</u>



<u>Check on Result:</u>

The result in Eq. (7) above can be checked by displacing the pole as shown by an amount $+i\epsilon$, and then taking the limit $\epsilon \to 0$. Again we wish to evaluate

$$I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0} = \lim_{\epsilon\to 0} \underbrace{\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-(x_0+i\epsilon)}}_{} \qquad (8)$$

$$\to \oint dz\, \frac{f(z)}{z-(x_0+i\epsilon)} = 2\pi i f(x_0+i\epsilon) \qquad (9)$$

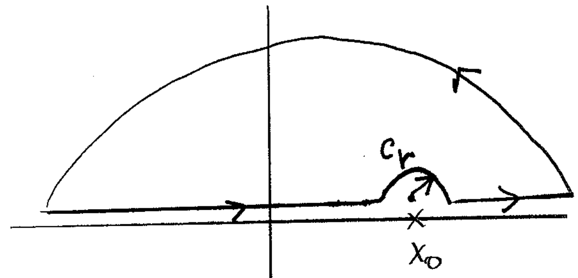
$$\to 2\pi i f(x_0)$$

DIRAC IDENTITY

$$\therefore\; 2\pi i f(x_0) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\, f(x)\left\{\frac{1}{x-x_0-i\epsilon}\right\} = \underbrace{P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0}}_{I} + \underbrace{i\pi\int_{-\infty}^{\infty} dx\, \delta(x-x_0) f(x)}_{i\pi f(x_0)} \qquad (10)$$

Hence $\boxed{I = 2\pi i f(x_0) - i\pi f(x_0) = i\pi f(x_0) \;\checkmark}$ (11)

We note that the same result can be obtained by deforming the original contour so as to exclude the pole on the real axis:



entire contour $\equiv C$

As before:
$$\oint_C dz \ldots = \underbrace{\int_{-\infty}^{x_0-r} + \int_{x_0+r}^{\infty}}_{P\int_{-\infty}^{\infty} dx\ldots} + \int_c (\text{small semicircle}) + \cancel{\int (\text{large semi-circle})}^{\,\to 0} \qquad (12)$$

NOTE CHANGE!

$$= 2\pi i\,[\text{Residues inside } C] \equiv 0 \qquad (13)$$

Hence from (12) & (13):
$$I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0} = -\oint_{C_r} \qquad (14)$$

Following the discussion on p.97 we have:

$$-\oint_{C_r} = -f(x_0)\int_{\pi}^{0} i\,d\theta = -f(x_0)\,i\,(0-\pi) = +i\pi f(x_0) \qquad (15)$$

Hence from (14) & (15):
$$\boxed{I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0} = +i\pi f(x_0)} \qquad (16)$$

This agrees with the results obtained previously in Eqs. (7) and (11).

<u>CONCLUSION:</u> There is often more than one way to evaluate a contour integral containing poles. We have seen 3 methods here:
a) <u>Include</u> the pole in the closed contour. b) <u>Exclude</u> the pole from the closed contour. c) <u>Displace</u> the pole off the real axis.

APPLICATION Evaluate $I_1 = P\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x}$

$$I_2 = P\int_{-\infty}^{\infty} dx\, \frac{\cos x}{x} \qquad (1)$$

Using the previous results we first note that

$$I_1 = \mathrm{Im}\left\{P\int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x}\right\} \;;\; I_2 = \mathrm{Re}\left\{P\int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x}\right\} \qquad (2)$$

Hence we evaluate

$$P\int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x} = \pi i\left[\text{Residue of pole at } x=0\right] \qquad (3)$$

$$= \pi i\left[e^{iz}\right]_{z=x=0} = \pi i \cdot 1$$

$$\therefore I_1 = \mathrm{Im}[i\pi] = \pi \Rightarrow \boxed{P\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} = \pi} \qquad (4)$$

$$I_2 = \mathrm{Re}\left[i\pi\right] = 0 \Rightarrow \boxed{P\int_{-\infty}^{\infty} dx\, \frac{\cos x}{x} = 0} \qquad (5)$$

COMMENTS:

1) The result in (4) is in agreement with the result previously obtained on p. CV-54(3) using Hilbert transform pairs.

2) The result in (5) follows by symmetry: The integrand is an odd function of x being integrated over a symmetric interval.

3) Despite appearances the integrand in (4) does not actually have a pole at $x=0$: Recall that $\sin x/x \to 1$ as $x \to 0$. Nonetheless this formalism applies.

4) When the contours in (1) and (2) are formed into closed contours in the u.h.p., the integral over the large semi-circle $\to 0$ by the usual arguments: $e^{iz} \underset{\text{u.h.p}}{\to} e^{i(x+iy)} \to e^{-y}e^{ix} \to 0$.

Collecting the previous results together:

$$A_n = \frac{1}{2\pi i}\oint dz' \frac{f(z')}{(z'-z_0)^{n+1}} = \frac{1}{2\pi i}\int (i\cancel{e^{i\theta}}d\theta)\frac{\cosh(2\cos\theta)}{e^{in\theta}\,\cancel{e^{i\theta}}} \qquad (9)$$

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \cosh(2\cos\theta)\underbrace{e^{-in\theta}}_{\cos n\theta + i\sin n\theta} \qquad (10)$$

odd function

$$\boxed{A_n = \frac{1}{\pi}\int_0^{\pi} d\theta \cosh(2\cos\theta)\cos n\theta} \qquad (11)$$

This can be evaluated by techniques that we develop next semester, either in terms of beta functions, or by using integral representations of Bessel functions. It can be shown that

$$A_{2n} = \sum_{m=0}^{\infty} \frac{1}{m!\,(m+2|n|)!} \quad ; \quad A_{2n+1} = 0 \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$$

<u>Another Example</u>: $f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} \Rightarrow A_n = \frac{1}{n!}$

This function has an <u>essential singularity</u> at $z=0$ (singularity cannot be removed by multiplying $f(z)$ by any <u>finite</u> power of $(z-z_0)$). One can formally carry out a Laurent expansion, but by uniqueness we must end up with the same expansion we get by simply expanding the exponential. We will use this observation later when evaluating contour integrals,

CLASSIFICATION OF SINGULARITIES

"Some singularities are worse than others!"

Whenever the function $\to \infty$ there is a singularity. For example all of the following have a singularity at $z=0$

$$\frac{1}{z}\ ,\quad \frac{1}{z^2}\ ,\quad e^{1/z}$$

Intuitively we feel that $1/z^2$ is a worse singularity than $1/z$, and that $e^{1/z}$ is the worst of all. This motivates us (for later purposes) to classify singularities, which can be done using the LAURENT expansion.

Write
$$f(z_0+z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

<u>Definition:</u> If $b_n = 0$ for $n \geq N+1$ $f(z_0+z)$ is said to have a pole of order N at z_0. Then the contribution from the negative powers is called the <u>Principal Part of $f(z)$:</u>

$$\left[\frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \frac{b_N}{z^N}\right] \quad ; \quad e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$$

pole of order 1   pole of order 2   essential singularity at $z=0$

As before, we can study the behavior of $f(z)$ at $z \to \infty$ by substituting $w = 1/z$, and then examining the function at $w=0$.

<u>Examples:</u> 1) $f(z)\ (=z)$ is not analytic at $z=\infty$ since $f(z) = f(1/w) = g(w)$ has a first order pole at $w=0$.

(2) e^z has an essential singularity at $z=\infty$ since $e^{1/w}$ has an essential singularity at ~~...~~ $w=0$.

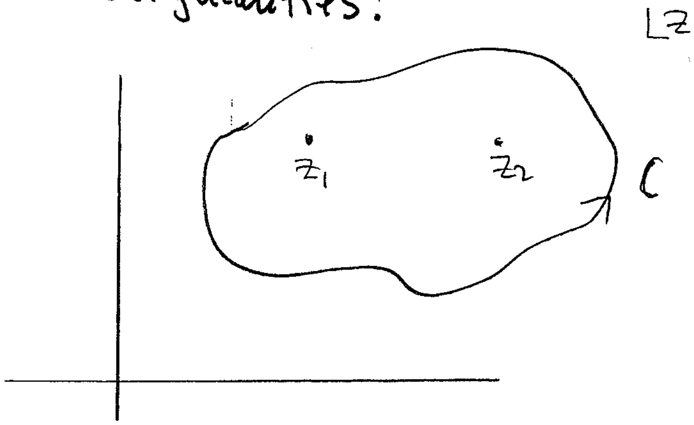
<u>Definition:</u> An <u>entire function</u> is one with no singularities in the finite part of the plane. [A singularity at ∞ is ok]

<u>Definition:</u> A function is <u>meromorphic</u> in some region R if it has no essential singularities in that region.

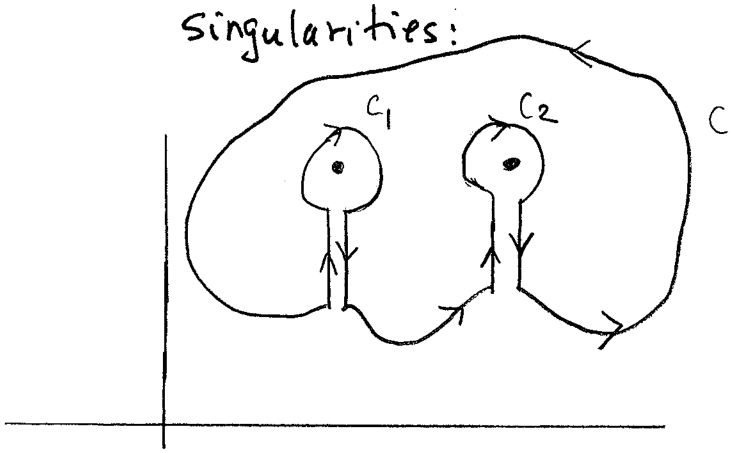
<u>Definition:</u> A <u>rational function</u> is one which is meromorphic in the entire complex plane. As the name suggests, it can be proved that such a function is the quotient of two polynomials.

THEORY OF COMPLEX INTEGRATION: "RESIDUE THEORY"

Problem: Evaluate $\oint_C dz\, f(z)$ where C is a contour enclosing singularities:



Method: Start by redrawing the contour so as to <u>exclude</u> the singularities:



C_1, C_2 are CW contours!

$C' = C + C_1 + C_2$ + cancelling pieces $\uparrow\downarrow$

Then since this contour does not enclose any singularities we have

$$\oint_{C'} dz\, f(z) = 0 = \oint_C dz\, f(z) + \oint_{-C_1} dz\, f(z) + \oint_{-C_2} dz\, f(z) \qquad (1)$$

Hence:

RESIDUE THEOREM

$$\boxed{\begin{gathered}\oint_C dz\, f(z) = \oint_{C_1} dz\, f(z) + \oint_{C_2} dz\, f(z) \equiv 2\pi i\,(\underbrace{R_1 + R_2}_{\text{Residues}}) \\ R_1 = \frac{1}{2\pi i}\oint_{C_1} dz\, f(z) \quad ; \quad R_2 = \oint_{C_2} dz\, f(z) \cdot \frac{1}{2\pi i}\end{gathered}} \qquad (2)$$