From Eq. (9) we then have:

$$I = -2i \oint_{|z|=1} dz \, \frac{1}{(z-\alpha)(z-\beta)} = -2i \oint_{|z|=1} dz \, \frac{(1/(z-\beta))}{(z-\alpha)} \tag{14}$$

$$= \oint_{|z|=1} dz \, \frac{\left[\frac{-2i}{z-\beta}\right]}{z-\alpha} = 2\pi i \; b_1(z=\alpha) = 2\pi i \left[\frac{-2i}{z-\beta}\right]_{z=\alpha} \tag{15}$$

$$\therefore \boxed{I = \frac{4\pi}{\alpha-\beta} = \frac{2\pi}{\sqrt{a^2-1}}} \tag{16}$$

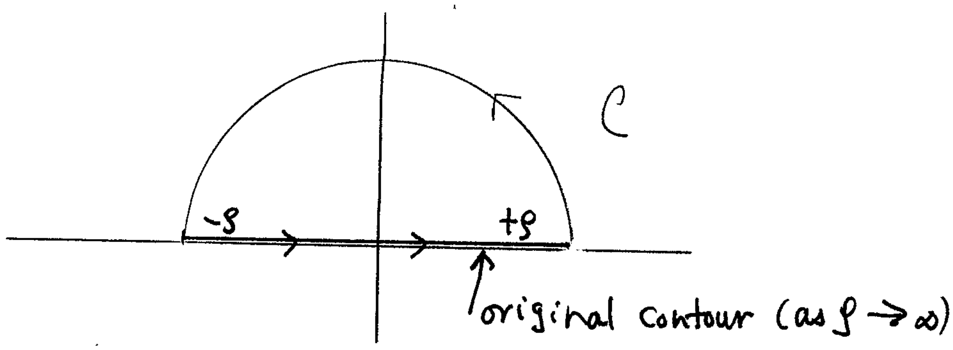
As anticipated, the integral is <u>real</u>, as it must be, but we have used complex integration (contour integration) to obtain this result.

(B) Consider next the following class of integrals

$$I = \int_{-\infty}^{\infty} dx\, R(x) \qquad \text{for example } I = \int_{-\infty}^{\infty} dx \frac{1}{(x^2+1)^2} \qquad (1)$$

$R(x)$ is a rational polynomial in x, subject to the restriction that it has no poles on the x-axis. [We will deal with the case of poles on the real axis later.] The integral in (1) exists if the degree of the polynomial in the denominator is at least 2 units higher than the degree of the numerator [We explain why below.]

Method: Begin by extending the contour to form a semi-circle as below:



$$\oint_C = \int_{-\infty}^{\infty} + \int_{\text{Semi-circle}} \Rightarrow \int_{-\infty}^{\infty} = \oint_C - \int_{\text{Semi-circle}} \qquad (2)$$

First evaluate $\oint_C = \oint dz\, R(z)$ → note that in this approach z becomes a complex variable in evaluating $\oint dz$

Hence $\oint dz\, R(z) = 2\pi i \{ \Sigma \text{ residues in upper half plane} \}$ (3)

$$\therefore \boxed{\oint dz\, R(z) \equiv 2\pi i \left\{ \sum_{y>0} \text{Residues}\left[R(z)\right] \right\}} \qquad (4)$$

Consider next $\int_{\text{Semi-circle}}$: As $\rho \to \infty$ we have

$$\left| \int_{\text{Semi-circle}} dz\, R(z) \right| \leq \int \text{const} \frac{|dz|}{\rho^2} = \int \text{Const} \frac{|\rho\, d\theta|}{\rho^2} \to \text{const} \int \frac{d\theta}{\rho} \to 0 \qquad (5)$$

here is where we invoke the assumption that the degree of the polynomial in the denominator is at least 2 greater than in the numerator.

Net result: $\int_{\text{Semi-circle}} \to 0$ under these conditions.

Hence finally:

$$\boxed{\int_{-\infty}^{\infty} dx\, R(x) = \oint_C dz\, R(z) = 2\pi i \sum_{\underbrace{y>0}_{\text{upper half plane}}} \text{Res}\left[R(z)\right]} \qquad (6)$$

Example:

$$I = \int_{-\infty}^{\infty} dx \frac{1}{(x^2+1)^2} = \oint_{\underset{\text{large Semi-circle}}{C}} dz \frac{1}{(z^2+1)^2} = \oint dz \frac{1}{[(z+i)(z-i)]^2} \qquad (7)$$

$$= \oint dz \frac{1}{(z+i)^2(z-i)^2}$$

← Only the root at $z=+i$ contributes, since the root at $z=-i$ is not inside the contour

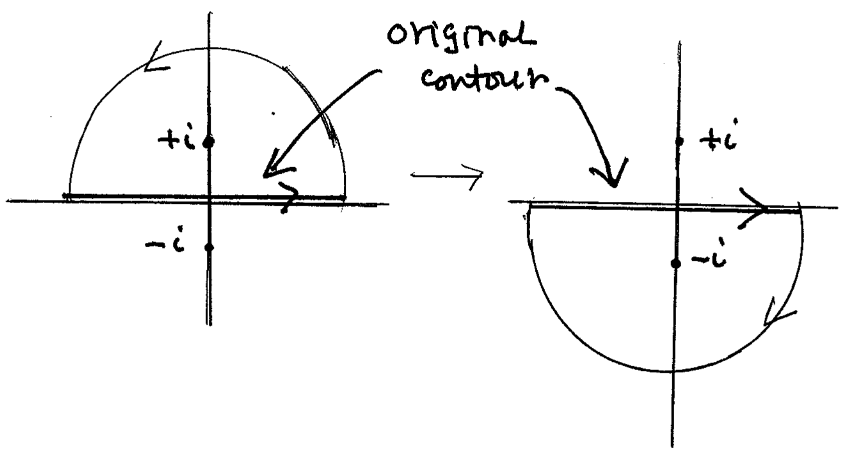
Hence $$I = \oint dz \frac{\left(1/(z+i)^2\right)}{(z-i)^2} = \frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{1}{z+i}\right)^2 \Big|_{z=i} = -\frac{4\pi i}{(z+i)^3}\Big|_{z=i} \qquad (8)$$

$$\therefore \boxed{I = -\frac{4\pi i}{(2i)^3} = \frac{\pi}{2} = \text{REAL}} \qquad (9)$$

COMMENTS:

(a) As a rough rule of thumb, if a real integral gives a result proportional to π, then it can be readily evaluated using contour integration, from which factors of 2π naturally arise.

(b) The original real integral was $\int_{-\infty}^{\infty} dx \cdots$, and we chose to form this into a closed contour by adding a semi-circle in the upper-half-plane (u.h.p). The natural question is what would have permitted us to arrive at the same result, had we closed the contour in the lower-half-plane?



$$I = \oint dz \, \frac{1}{(z+i)^2(z-i)^2}$$ ← now only the root at $z = -i$ contributes since $z = +i$ is outside the contour (10)

$$\text{Hence } I = \oint dz \, \frac{\left(1/(z-i)^2\right)}{(z+i)^2} = -\frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{1}{z-i}\right)^2\Bigg|_{z=-i} = +\frac{4\pi i}{(z-i)^3}\Bigg|_{z=-i} \quad (11)$$

CW contour!

$$= \frac{+4\pi i}{(-2i)^3} = \frac{\pi}{2} \quad (12)$$

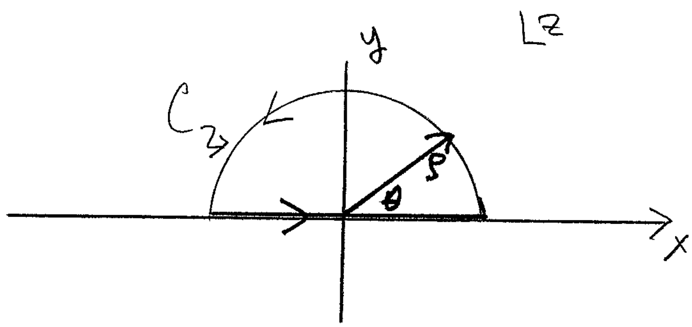
SAME AS BEFORE! (See (9))

CONCLUSION: $\int_{-\infty}^{\infty}$ can be made into a closed contour by adding a semi-circle in either the u.h.p or the l.h.p. EITHER WAY ⇒ SAME RESULT!

(c) The next class of integrals we consider is

$$I=\int_{-\infty}^{\infty}dx\, R(x)\, e^{ix} = \int_{-\infty}^{\infty}dx\, R(x)\cos x + i\int_{-\infty}^{\infty}dx\, R(x)\sin x \qquad (1)$$

where $R(x)$ is a polynomial in x as before. The condition for the existence of these integrals is now that $|R(z)|\to 0$ uniformly in θ as $\rho\to\infty$. [JORDAN'S LEMMA]



$\rho\to\infty$ is understood

Following the previous discussion we can write

$$\boxed{I=\int_{-\infty}^{\infty}dx\, R(x)\, e^{ix} = \oint_C dz\, R(z)\, e^{iz} = 2\pi i \sum_{y>0} \mathrm{Res}\left[R(z)\, e^{iz}\right]} \qquad (2)$$

Comments: For this class of integrals the contour <u>must</u> be closed in the u.h.p. only (not the l.h.p) to ensure that in the vicinity of the imaginary axis the integrand is a damped exponential:

$$e^{iz} \xrightarrow{\text{u.h.p.}} e^{i(x+iy)} = e^{ix}e^{-y} \xrightarrow{\rho\to\infty} 0 \qquad (3)$$

Had the contour been closed in the l.h.p. we would have a diverging factor in the integrand:

$$e^{iz} \xrightarrow{\text{l.h.p.}} \cancel{e^{ix}}\, e^{i(-iy)} = e^{ix}e^{+y} \xrightarrow{\rho\to\infty} \infty \qquad (4)$$

The situation is reversed if the integrand contains the factor e^{-iz}.

Example: $I = \int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k^2+\mu^2}$ $\quad r = \text{fixed} > 0$ (5)

Let $x = kr \Rightarrow I = r \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x^2+\Lambda^2}$ $\quad \Lambda = \mu r$ (6)

Then from 93(2): $I = r \oint_c dz \, \frac{e^{iz}}{z^2+\Lambda^2} = r \cdot 2\pi i \sum_{\text{Res } y>0} \left[\frac{e^{iz}}{z^2+\Lambda^2} \right]$ (7)

The poles are determined by $\frac{e^{iz}}{z^2+\Lambda^2} = \frac{e^{iz}}{(z+i\Lambda)(z-i\Lambda)} = \frac{e^{iz}/(z+i\Lambda)}{z-i\Lambda}$ (8)

this is in u.h.p.

Since the only pole in the u.h.p. is $z = +i\Lambda$ we find:

$$I = r \cdot 2\pi i \left[\frac{e^{iz}}{z+i\Lambda} \right]_{z=i\Lambda} = 2\pi i r \left[\frac{e^{i(i\Lambda)}}{2i\Lambda} \right] = \frac{\pi r}{\Lambda} e^{-\Lambda} \quad (9)$$

$$\therefore \boxed{I = \frac{\pi r}{\Lambda} e^{-\Lambda} = \pi \frac{e^{-\mu r}}{\mu}} \quad (10)$$

Comments: When given an integral of the form $I_1 = \int_{-\infty}^{\infty} dx \, \frac{\cos x}{x^2+a^2}$ (11)

We can evaluate it in 2-ways:

(12)

(a) $I_1 = \text{Re} \oint_c dz \, \frac{e^{iz}}{z^2+a^2} = \text{Re}\left[\pi \frac{e^{-a}}{a} \right] = \pi \frac{e^{-a}}{a}$ using (7) & (10)

Note incidentally that $I_2 = \int_{-\infty}^{\infty} dx \, \frac{\sin x}{x^2+a^2} = \text{Im} \oint_c dz \, \frac{e^{iz}}{z^2+a^2}$ (13)

$= \text{Im}\left[\pi \frac{e^{-a}}{a} \right] = 0$ (14)

This makes sense, since I_2 is the integral of an odd function over a symmetric interval.

(b) The other way of writing the integral is:

$$\cos x = \frac{1}{2}(e^{ix}+e^{-ix}) \Rightarrow I = \frac{1}{2}\oint dz\,(e^{iz}+e^{-iz})\frac{1}{z^2+a^2} \qquad (15)$$

However, now we have to treat e^{iz} and e^{-iz} separately to ensure convergence:

$$I = \underbrace{\frac{1}{2}\oint_C dz\,\frac{e^{iz}}{(z+ia)(z-ia)}}_{I_3 = \text{Close in u.h.p}} + \underbrace{\frac{1}{2}\oint_C dz\,\frac{e^{-iz}}{(z+ia)(z-ia)}}_{I_4 = \text{close in l.h.p.}} \qquad (16)$$

$$I_3 = \frac{1}{2}I_1 = \frac{1}{2}\pi\frac{e^{-a}}{a} \qquad (17)$$

CW contour!

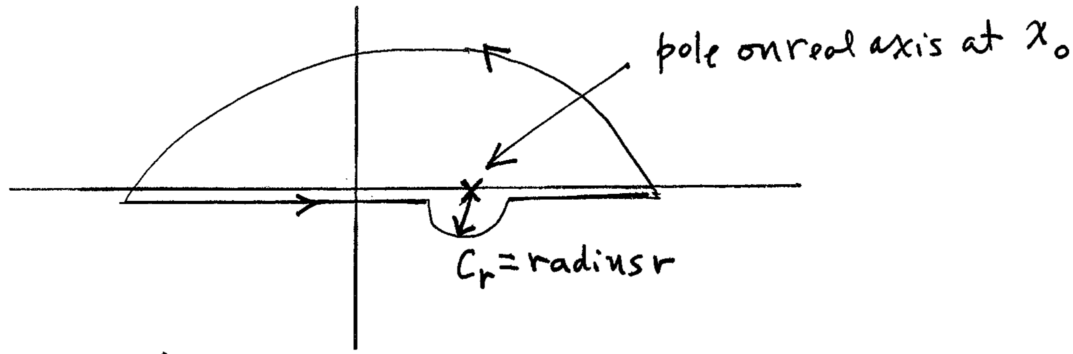
$$I_4 = \frac{1}{2}\oint_{\substack{C \\ \text{l.h.p.}}} dz\,\frac{e^{-iz}/(z-ia)}{z+ia} = -2\pi i\frac{1}{2}\left[\frac{e^{-iz}}{z-ia}\right]_{z=-ia} \qquad (18)$$

$$I_4 = -\pi i\,\frac{e^{-i(-ia)}}{-2ia} = \frac{\pi}{2}\frac{e^{-a}}{a} \qquad (19)$$

$$\therefore \boxed{I = I_3 + I_4 = \frac{1}{2}\pi\frac{e^{-a}}{a} + \frac{1}{2}\pi\frac{e^{-a}}{a} = \pi\frac{e^{-a}}{a}} \qquad (20)$$

This is the same result obtained in (12) using the previous method.

CONTOUR INTEGRALS INVOLVING POLES ON THE REAL AXIS



Consider $I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x - x_0} \longrightarrow \oint_C dz\, \frac{f(z)}{z - x_0}$ (1)

Principal value

$$\oint_C dz \ldots = \int_{-\infty}^{x_0 - r} + \int_{x_0 + r}^{\infty} + \oint_{C_r} (\text{small semicircle}) + \int (\text{large semicircle}) \nearrow 0 \quad (2)$$

assume this to hold

$\longrightarrow 2\pi i \left[\text{Residue of pole at } x_0\right] = 2\pi i f(x_0)$ (3)

Hence $\underbrace{\int_{-\infty}^{x_0 - r} \ldots + \int_{x_0 + r}^{+\infty} \ldots}_{\equiv P\int_{-\infty}^{\infty} dx \ldots} = 2\pi i f(x_0) - \oint_{C_r} dz\, \frac{f(z)}{z - x_0}$ (4)

As the radius r of the small semi-circle decreases the l.h.s. of (4) becomes the integral I that we are trying to evaluate. Hence

$$I = 2\pi i f(x_0) - \oint_{C_r} dz\, \frac{f(z)}{z - x_0} \overset{r \to 0}{=} 2\pi i f(x_0) - f(x_0) \oint dz\, \frac{1}{z - x_0} \quad (5)$$

along semicircle

Along the semi-circle: $z - x_0 = re^{i\theta}$; $dz = ire^{i\theta} d\theta$

$$\oint dz \ldots \longrightarrow \int_{-\pi}^{0} d\theta \cdot \frac{ire^{i\theta}}{re^{i\theta}} = i\int_{-\pi}^{0} d\theta = +\pi i \quad (6)$$