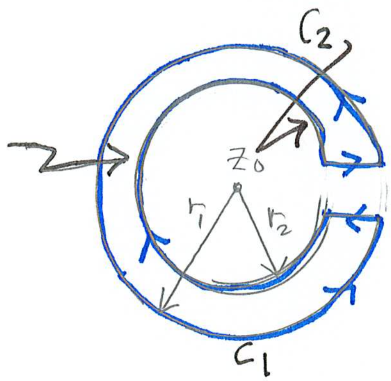
TAYLOR & LAURENT SERIES

$f(z)$ is analytic here



LAURENT'S THEOREM (GENERALIZATION OF TAYLOR EXPANSION)

If $f(z)$ is analytic in the region between C_1 and C_2 then $f(z)$ can be expanded in this region as:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n \; ; \quad A_n = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}} \qquad (1)$$

note presence of negative powers

any closed contour circling z_0

Comments:

(1) The proof is direct, but will not be given in class. [See BYRON/FULLER p. 349]

(2) The Laurent theorem allows for the presence of singularities by including negative powers of $(z'-z_0)$.

(3) For a function $f(z)$ which is analytic everywhere we have:

(a) for $n \leq -1$ $\quad A_n = \frac{1}{2\pi i}\oint_C dz' f(z')(z'-z_0)^{-n-1} \qquad (2)$

$$= \frac{1}{2\pi i}\oint_C dz' \underbrace{f(z')\underbrace{(z'-z_0)^{|n|-1}}_{1 \text{ or positive power of } (z'-z_0)}}_{\text{analytic}} \equiv 0 \qquad (3)$$

(b) For $n \geq 0$ we have

$$A_n = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0) \quad \Big\} \text{ CAUCHY INTEGRAL FORMULA} \quad (4)$$

Hence altogether, if $f(z)$ is analytic

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z-z_0)^n} \quad (5)$$

TAYLOR'S THEOREM

(4) When a singular function can be expanded by inspection into various inverse powers, then this must be the result obtained by formally evaluating $\oint_C$, since an expansion about a singular point is unique. Example: [B/F p.352]

$$f(z) = \frac{z^3+2z^2+4}{(z-1)^3} = 1 + \frac{5}{z-1} + \frac{7}{(z-1)^2} + \frac{7}{(z-1)^3} \quad (6)$$

Examples of a Laurent Expansion:

$f(z) = \cosh\left(z+\frac{1}{z}\right)$; this is analytic everywhere except at $z=0$.

Choose C_2 = arbitrarily small circle about origin, and C_1 to be arbitrarily large. Then choose C = unit circle about the origin:

$$z'-z_0 = 1\,e^{i\theta} \qquad dz' = i\,e^{i\theta}\,d\theta \quad (7)$$

$$z'+\frac{1}{z'} = e^{i\theta}+\frac{1}{e^{i\theta}} = e^{i\theta}+e^{-i\theta} = 2\cos\theta$$

$$\frac{1}{z'^{n+1}} \equiv \frac{1}{(z'-z_0)^{n+1}} = e^{-in\theta}\,e^{-i\theta} \quad (8)$$

Collecting the previous results together:

$$A_n = \frac{1}{2\pi i}\oint dz' \frac{f(z')}{(z'-z_0)^{n+1}} = \frac{1}{2\pi i}\int (i\cancel{e^{i\theta}}\, d\theta)\,\frac{\cosh(2\cos\theta)}{e^{in\theta}\,\cancel{e^{i\theta}}} \qquad (9)$$

$$A_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\theta\, \cosh(2\cos\theta)\,\underbrace{e^{-in\theta}}_{\cos n\theta\, \pm i\sin n\theta} \qquad (10)$$

odd function

$$\boxed{A_n = \frac{1}{\pi}\int_0^{\pi} d\theta\, \cosh(2\cos\theta)\cos n\theta} \qquad (11)$$

This can be evaluated by techniques that we develop next semester, either in terms of beta functions, or by using integral representations of Bessel functions. It can be shown that

$$A_{2n} = \sum_{m=0}^{\infty} \frac{1}{m!\,(m+2|n|)!} \quad ; \quad A_{2n+1} = 0 \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$$

<u>Another Example</u>: $f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} \Rightarrow A_n = \frac{1}{n!}$

This function has an <u>essential singularity</u> at $z=0$ (singularity cannot be removed by multiplying $f(z)$ by any <u>finite</u> power of $(z-z_0)$).
One can formally carry out a Laurent expansion, but by uniqueness we must end up with the same expansion we get by simply expanding the exponential. We will use this observation later when evaluating contour integrals.

CLASSIFICATION OF SINGULARITIES

"Some Singularities are worse than others!"

Whenever the function $\to \infty$ there is a singularity. For example all of the following have a singularity at $z=0$

$$\frac{1}{z}, \quad \frac{1}{z^2}, \quad e^{1/z}$$

Intuitively we feel that $1/z^2$ is a worse singularity than $1/z$, and that $e^{1/z}$ is the worst of all. This motivates us (for later purposes) to classify singularities, which can be done using the LAURENT expansion.

Write
$$f(z_0+z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

Definition: If $b_n = 0$ for $n \geq N+1$ $f(z_0+z)$ is said to have a pole of order $\cancel{N}$ n at z_0. Then the contribution from the negative powers is called the Principal Part of $f(z)$:

$$\left[\frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \frac{b_N}{z^N}\right] \quad ; \quad e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$$

pole of order 1   pole of order 2   essential singularity at $z=0$

As before, we can study the behavior of $f(z)$ at $z \to \infty$ by substituting $w = 1/z$, and then examining the function at $w=0$.

Examples: 1) $\underbrace{f(z)}_{(=z)}$ is not analytic at $z=\infty$ since $f(z) = f(1/w) = g(w)$ has a first order pole at $w=0$.

(2) e^z has an essential singularity at $z=\infty$ since $e^{1/w}$ has an essential singularity at $w=0$.

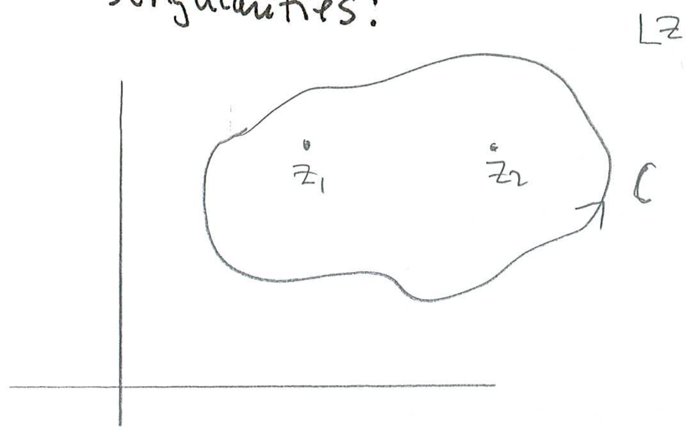
Definition: An entire function is one with no singularities in the finite part of the plane. [A singularity at ∞ is ok]

Definition: A function is meromorphic in some region R if it has no essential singularities in that region.

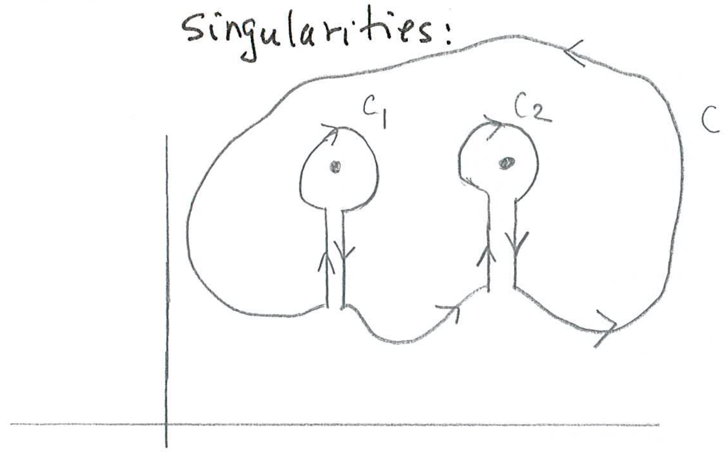
Definition: A rational function is one which is meromorphic in the entire complex plane. As the name suggests, it can be proved that such a function is the quotient of two polynomials.

THEORY OF COMPLEX INTEGRATION: "RESIDUE THEORY"

Problem: Evaluate $\oint_C dz\, f(z)$ where C is a contour enclosing singularities:



Method: Start by redrawing the contour so as to exclude the singularities:



C_1, C_2 are CW contours!

$C' = C + C_1 + C_2$ + cancelling pieces $\uparrow\downarrow$

Then since this contour does not enclose any singularities we have

$$\oint_{C'} dz\, f(z) = 0 = \oint_C dz\, f(z) + \oint_{-C_1} dz\, f(z) + \oint_{-C_2} dz\, f(z) \qquad (1)$$

Hence:

RESIDUE THEOREM

$$\boxed{\oint_C dz\, f(z) = \oint_{C_1} dz\, f(z) + \oint_{C_2} dz\, f(z) \equiv 2\pi i\,(R_1 + R_2) \qquad R_1 = \frac{1}{2\pi i}\oint_{C_1} dz\, f(z)\ ;\ R_2 = \oint_{C_2} dz\, f(z)\cdot\frac{1}{2\pi i}} \qquad (2)$$

Residues

At this stage we can evaluate $\oint_C f(z)\,dz$ by evaluating $\oint_{C_i} dz\, f(z)$ around C_i any way that we want.

In fact we can develop a simple formula for evaluating the residues, as follows: Suppose there is a singularity at z_0 then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} \tag{3}$$

$$a_n = \frac{1}{2\pi i}\oint_{C_0} dz' \frac{f(z')}{(z'-z_0)^{n+1}} \quad ; \quad b_n = \frac{1}{2\pi i}\oint_{C_0} dz' \frac{f(z')}{(z'-z_0)^{-n+1}} \tag{4}$$

$$\therefore \oint_{C_0} f(z)\,dz \equiv 2\pi i R_0 = \sum_{n=0}^{\infty} a_n \underbrace{\oint_{C_0} dz\,(z-z_0)^n}_{\substack{\parallel \\ 0 \text{ by Cauchy theorem}}} + \sum_{n=1}^{\infty} b_n \oint_{C_0} dz \frac{1}{(z-z_0)^n} \tag{5}$$

Let C_0 = circle of radius r_0 around z_0. Then

$$(z-z_0)^n = r_0^n e^{in\theta} \qquad dz = i r_0 e^{i\theta} d\theta \tag{6}$$

$$\oint_{C_0} \cdots = \int_0^{2\pi} \left(r_0^{-n} e^{-in\theta}\right)(i r_0) e^{i\theta} d\theta = i r_0^{-n+1} \int_0^{2\pi} d\theta\, e^{-i(n-1)\theta} \tag{7}$$

For $n>1$: $$\int_0^{2\pi} \cdots = \frac{1}{-i(n-1)} e^{-i(n-1)\theta}\Big|_0^{2\pi} = 0 \tag{8}$$

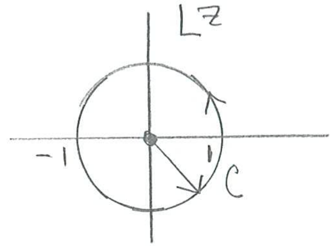
For $n=1$: $$\int_0^{2\pi} \cdots = i(r_0)^{-1+1} \int_0^{2\pi} d\theta = 2\pi i \tag{9}$$

Hence (5) & (9) $\Rightarrow$ $$\boxed{\oint_{C_0} dz\, f(z) = 2\pi i\, b_1 \equiv 2\pi i\, R(z=z_0) \;\Rightarrow\; R(z=z_0) = b_1 = \text{coefficient of } \frac{1}{z-z_0}} \tag{10}$$

Applications of Residue Theory:

[1] $I = \oint_C dz\, e^{1/z}$; C = Unit circle about origin

IMPORTANT! DRAW CONTOUR WITH SINGULARITIES

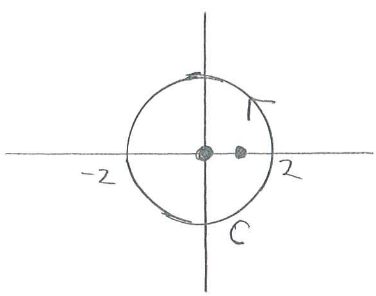


this is $\equiv b_1$ = coeff of $\frac{1}{z-0}$

Solution: $$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \frac{1}{(z-0)} + \frac{1}{2!\,(z-0)^2} + \cdots \qquad (11)$$

Hence by inspection $$\boxed{\oint_C dz\, f(z) \equiv 2\pi i (b_1) = 2\pi i \cdot 1 = 2\pi i} \qquad (12)$$

[2] $$I = \oint_C dz\, \frac{(5z-2)}{z(z-1)} \quad ; \quad C = \text{circle } |z| = 2 \qquad (13)$$



This contour surrounds 2 poles: at $z=0$, $z=1$

$$\oint_C dz\, f(z) = 2\pi i \left[\text{Residues } \underline{\text{inside}} \text{ Contour}\right]$$
$$= 2\pi i \left[b_1(z=0) + b_1(z=1)\right] \qquad (14)$$

At this stage we can carry out a Laurent expansion about $z=0$, and then about $z=1$. However, this can here be done <u>by inspection</u>

At $z=0$: Write $f(z)$ as $$f(z) = \frac{5z-2}{z(z-1)} = \frac{(5z-2)/(z-1)}{z} \qquad (15)$$

Near $z=0$ the function $f(z)$ behaves as $$f(z) \xrightarrow{z=0} \frac{(-2)/(-1)}{z} = \frac{2}{z} \qquad (16)$$

Hence <u>by uniqueness</u> <u>this must be the result of a formal Laurent expansion</u>. Hence $\boxed{b_1(z=0) = 2}$ (17)

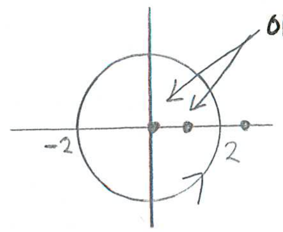
Similarly in the vicinity of $z=1$ we can write

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{(5z-2)/z}{z-1} \xrightarrow{z=1} \frac{3}{z-1} \Rightarrow \boxed{b_1(z=1) = 3} \quad (18)$$

Hence via the residue theorem: $\boxed{\oint_C dz\, f(z) = 2\pi i\,(2+3) = 10\pi i}$ (19)

Comment: There would have been the same number of residues had we considered the function

$$f(z) = \frac{5z-2}{z(z-1)(z-3)} \quad (20)$$



only these contribute

integrated along the same contour $|z|=2$. However, the residues at the two poles would have been different.

[3] $$I = \oint_C dz\, \frac{(5z-2)}{z(z-1)^3} \qquad C = \text{circle } |z| = 2 \quad (21)$$

By the residue theorem we have $\oint_C \ldots = \underbrace{\oint_{C_0} \ldots}_{\substack{z=0\\ \text{pole}}} + \underbrace{\oint_{C_1} \ldots}_{\substack{z=1\\ \text{pole}}}$ (22)

$$\therefore \oint_C \ldots = \oint_{C_0} dz\, \frac{(5z-2)/(z-1)^3}{z} + \oint_{C_1} dz\, \frac{(5z-2)/z}{(z-1)^3} \quad (23)$$

$$\oint_{C_0} \ldots = \oint_{C_0} dz\, \frac{(5z-2)/(z-1)^3}{z-0} = 2\pi i \left[b_1(z=0) = (-2)/(-1)^3 \right] = 4\pi i \quad (24)$$

$$\oint_{C_1} dz \cdots = \oint_{C_1} dz \left\{ \frac{(5z-2)/z}{(z-1)^3} \right\} \qquad (25)$$

Comment: At this stage we could expand the entire integrand in a Laurent series and pick out the coefficient $b_1(z=1)$. However, by virtue of the same uniqueness argument that we gave on p. CV-83 we can use the Cauchy derivative formula, which is a more convenient way to get the same result. Recall:

$$\boxed{\oint_C dz\, g(z) = \oint_C dz \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)} \qquad (26)$$

In (25) & (26) $f(z) = (5z-2)/z = (5-2/z)$, $n=2$ $\Rightarrow$

$$f'(z) = \frac{d}{dz}\left(5 - \frac{2}{z}\right) = \frac{2}{z^2}, \qquad f''(z) = -\frac{4}{z^3} \qquad (27)$$

$$\text{Hence } \oint_{C_1} dz \cdots = \frac{2\pi i}{2!}\left[-\frac{4}{z^3}\right]_{z=1} = -4\pi i \qquad (28)$$

Combining Eqs. (24) & (28) we see that

$$\oint_{C \Rightarrow |z|=2} dz \frac{(5z-2)}{z(z-1)^3} = \oint_{C_0} dz \cdots + \oint_{C_1} dz \cdots = \Big[\underbrace{+4\pi i}_{z=0} \; \underbrace{-4\pi i}_{z=1}\Big] = 0$$

This is an interesting example because it shows that a function can be non-analytic (has poles) and yet give $\oint_C dz \cdots = 0$ for some contour C, by virtue of cancellations.

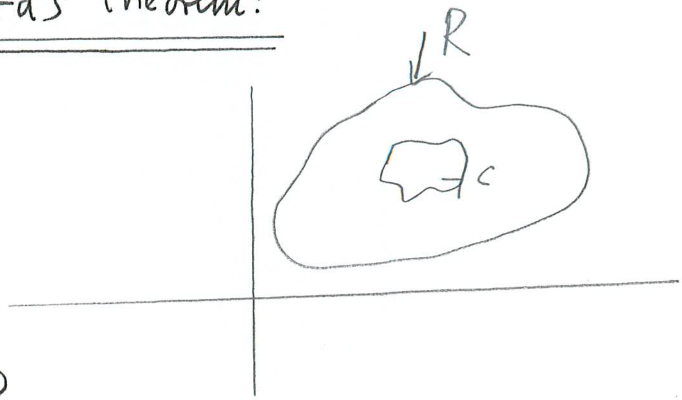
See CV-45

Connection to Morera's Theorem:

Cauchy's Theorem

If $f(z)$ is analytic in R then $\oint_C dz\, f(z) = 0$

for any C.



Morera's Theorem (Converse of Cauchy theorem)

If $f(z)$ is continuous in R and if $\oint_C dz\, f(z) = 0$ for any C* in R then $f(z)$ is analytic in R.

In the previous example $f(z) = \dfrac{5z-2}{z(z-1)^3}$ is clearly not analytic in the region defined by the circle $|z|=2$. Nonetheless $\oint dz\, f(z) = 0$. This does not contradict Morera's theorem, which holds that $f(z)$ is analytic only if $\oint_C dz\, f(z)$ for any C*! In the present example $\oint dz\, f(z) \neq 0$ if C is the circle $|z| = 0.5$

So the conditions for Morera's theorem are not met.

Two Views on Evaluating Integrals with Higher-Order Poles

Return to the previous problem:

$$I = \oint_C dz \, \frac{5z-2}{z(z-1)^3} \qquad C = \text{circle } |z| = 2 \qquad (1)$$

Residue theorem: $$I = \oint_{C_0} dz \cdots + \oint_{C_1} dz \cdots \qquad (2)$$

Since $\oint_{C_1} dz \ldots$ involves a higher-order ($N=3$) pole we previously used the Cauchy formula:

$$\frac{1}{2\pi i}\oint_{C_1} dz \, g(z) = \frac{1}{2\pi i}\oint_{C_1} dz \, \frac{f(z)}{(z-z_1)^{n+1}} = \frac{1}{n!} f^{(n)}(z_1) \qquad (3)$$

This then gives:

$$\boxed{\oint_{C_1} dz \, \frac{f(z)}{(z-z_1)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_1)} \qquad (4)$$

However, we have previously shown [p. CV-81,82] that we can evaluate the integral in (3) by writing

coefficient of $1/(z-z_1)$

$$\boxed{\frac{1}{2\pi i}\oint_{C_1} dz \, g(z) = b_1 = \text{coefficient of } \left(1/z-z_1\right) \text{ in the Laurent expansion of } g(z)} \qquad (5)$$

Q: What is the connection between the use of (4) and (5) to evaluate $g(z)$?

A: When $g(z) = f(z)/(z-z_1)^{n+1}$, with $f(z)$ analytic at z_1 we can expand $f(z)$ as follows:

$$f(z) = f(z_1) + (z-z_1) f^{(1)}(z_1) + \frac{1}{2!}(z-z_1)^2 f^{(2)}(z_1) + \cdots + (z-z_1)^{n-1}\frac{1}{(n-1)!} f^{(n-1)}(z_1)$$
$$+ (z-z_1)^n \frac{1}{n!} f^{(n)}(z_1) + \cdots \qquad (6)$$

It follows from (6) that the integrand in (5) is given by

$$g(z) = \frac{f(z)}{(z-z_1)^{n+1}} = \frac{f(z_1)}{(z-z_1)^{n+1}} + \frac{f^{(1)}(z_1)}{(z-z_1)^{n}} + \frac{1}{2!}\frac{f^{(2)}(z_1)}{(z-z_1)^{n-1}} + \dots$$

$$+ \frac{1}{(z-z_1)}\left[\frac{1}{n!} f^{(n)}(z_1)\right] + \frac{1}{(n+1)!} f^{(n+1)}(z_1) + \dots \qquad (7)$$

Now the general result that the residue of $g(z)$ at z_0 is just the coefficient b_1 of $1/(z-z_0)$ then gives

$$\boxed{b_1 = \left[\frac{1}{n!} f^{(n)}(z_1)\right] \Rightarrow \oint_{C_1} dz\, g(z) = 2\pi i\, b_1 = \frac{2\pi i}{n!} f^{(n)}(z_1)} \qquad (8)$$

We thus the result in Eq. (4) obtained previously using the Cauchy formula.

<u>CONCLUSION:</u> When evaluating $\oint_{C_0} dz\, g(z) = \oint_{C_0} dz \frac{f(z)}{(z-z_0)^{n+1}}$

<u>We can use the Cauchy formula in (4) directly</u> since this is equivalent to doing a Laurent expansion of $g(z)$ and finding the coefficient b_1 of $1/(z-z_0)$.

EVALUATION OF REAL INTEGRALS VIA CONTOUR INTEGRATION

Ⓐ Let R be a rational function of $\sin\theta, \cos\theta, \sin^2\theta, \cos^2\theta, \ldots$

For example:

$$R(\cos\theta, \sin\theta) \equiv R = \frac{a_1\cos\theta + a_2\cos^2\theta + \ldots + a_n\cos^n\theta + b_1\sin\theta + b_2\sin^2\theta + \ldots}{c_1\cos\theta + c_2\cos^2\theta + \ldots + c_n\cos^n\theta + d_1\sin\theta + d_2\sin^2\theta + \ldots} \quad (1)$$

By using contour integration we can evaluate integrals of the form

$$I = \int_0^{2\pi} d\theta \; R(\cos\theta, \sin\theta) \quad (2)$$

Method: Let $\boxed{z = 1\, e^{i\theta}} \Rightarrow dz = i e^{i\theta} d\theta = i z\, d\theta \Rightarrow \boxed{d\theta = -i\dfrac{dz}{z}}$ (3)

Then: $$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad (4)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Then: $$\boxed{\int_0^{2\pi} d\theta\, R(\cos\theta, \sin\theta) = \oint_{C = \text{Unit Circle}} \left(-i\frac{dz}{z}\right) R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \quad (5)}$$

Example: $$I = \int_0^{2\pi} d\theta \frac{1}{a + \cos\theta} \qquad \underset{\sim\sim}{a > 1} \quad (6)$$

Note to begin that this is a real integral, which could in principle be evaluated using real integration. Nonetheless it is advantageous to carry out this integral via contour integration using (5).

Combining (3)–(6) we have

$$I = -i \oint_{|z|=1} \frac{dz}{z} \frac{1}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)} = -i \oint_{|z|=1} \frac{2\,dz}{z^2+2az+1} \qquad (7)$$

The poles will occur at the roots of $z^2+2az+1=0$ which are given by

$$\alpha = -a+\sqrt{a^2-1} \qquad \beta = -a-\sqrt{a^2-1} \qquad (8)$$

$$\therefore\ I = -2i \oint_{|z|=1} dz \frac{1}{(z-\alpha)(z-\beta)} \qquad (9)$$

To evaluate I we must determine which of these poles (if either!) lies within the contour $|z|=1$. Since $a>1$ it follows immediately that $\beta<-1 \Rightarrow |\beta|>1$. Hence the pole at $z=\beta$ lies outside the contour $|z|=1$, and thus does not contribute to the integral.

For α we can show that this pole does fall within $|z|=1$. To see this

$$a>1 \Rightarrow 2(1+a) = 1+(1+2a) > 0 \Rightarrow 1+2a > -1 \qquad (10)$$

$$\text{Adding } a^2 \text{ to both sides: } 1+2a+a^2 > -1+a^2 = a^2-1 \qquad (11)$$

$$\text{Take } \sqrt{} \text{ of both sides of (10)} \Rightarrow \underbrace{\sqrt{(1+2a+a^2)}}_{1+a} > \sqrt{a^2-1} \qquad (12)$$

$$\text{So } 1+a > \sqrt{a^2-1} \Rightarrow 1 > \underbrace{-a+\sqrt{a^2-1}}_{\alpha} \qquad (13)$$

$\therefore$ Finally, $\boxed{|\alpha|<1}$ $\rightarrow$ to ensure that $z=\alpha$ was inside the unit circle we took $a>1$.

This can also be seen by noting that for $a>1 \Rightarrow 1<a<\infty$

$\alpha = -a+\sqrt{a^2-1}$ varies between $\alpha=0\ (a=\infty)$ and $\alpha=-1\ (a=1)$. Hence α must be in the range $0<\alpha<-1 \Rightarrow |\alpha|<1$.

From Eq. (9) we then have:

$$I = -2i \oint_{|z|=1} dz \, \frac{1}{(z-\alpha)(z-\beta)} = -2i \oint_{|z|=1} dz \, \frac{(1/(z-\beta))}{(z-\alpha)} \qquad (14)$$

$$= \oint_{|z|=1} dz \, \frac{\left[\frac{-2i}{z-\beta}\right]}{z-\alpha} = 2\pi i \; b_1(z=\alpha) = 2\pi i \left[\frac{-2i}{z-\beta}\right]_{z=\alpha} \qquad (15)$$

$$\therefore \boxed{I = \frac{4\pi}{\alpha-\beta} = \frac{2\pi}{\sqrt{a^2-1}}} \qquad (16)$$

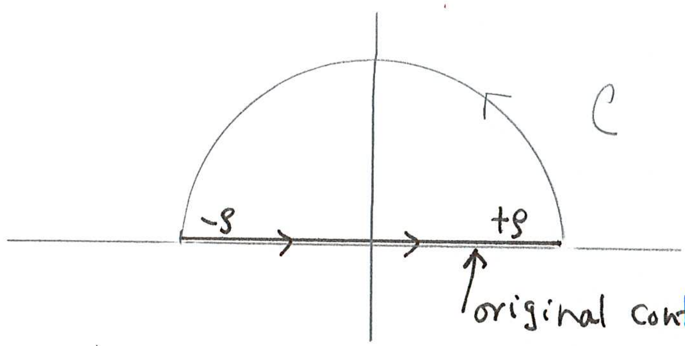
As anticipated, the integral is <u>real</u>, as it must be, but we have used complex integration (contour integration) to obtain this result.

(B) Consider next the following class of integrals

$$I = \int_{-\infty}^{\infty} dx\, R(x) \qquad \text{for example } I = \int_{-\infty}^{\infty} dx\, \frac{1}{(x^2+1)^2} \qquad (1)$$

$R(x)$ is a rational polynomial in x, subject to the restriction that it has no poles on the x-axis. [We will deal with the case of poles on the real axis later.] The integral in (1) exists if the degree of the polynomial in the denominator is at least 2 units higher than the degree of the numerator [We explain why below.]

Method: Begin by extending the contour to form a semi-circle as below:



$$\oint_C = \int_{-\infty}^{\infty} + \int_{\text{Semi-circle}} \Rightarrow \int_{-\infty}^{\infty} = \oint_C - \int_{\text{Semi-circle}} \qquad (2)$$

First evaluate $\oint_C = \oint dz\, R(z)$ → note that in this approach z becomes a complex variable in evaluating $\oint dz$

Hence $\oint dz\, R(z) = 2\pi i \{ \Sigma \text{ residues in upper half plane} \}$ (3)

$$\therefore \boxed{\oint dz\, R(z) = 2\pi i \left\{ \sum_{y>0} \text{Residues}\,[R(z)] \right\}} \quad (4)$$

Consider next $\int_{\text{Semi-circle}}$: As $\rho \to \infty$ we have

$$\left| \int_{\text{Semi-circle}} dz\, R(z) \right| \leq \int \text{const} \frac{|dz|}{\rho^2} = \int \text{Const} \frac{|\rho\, d\theta|}{\rho^2} \to \text{const} \int \frac{d\theta}{\rho} \to 0 \quad (5)$$

here is where we invoke the assumption that the degree of the polynomial in the denominator is at least 2 greater than in the numerator.

Net result: $\int_{\text{Semi-circle}} \to 0$ under these conditions.

Hence finally:

$$\boxed{\int_{-\infty}^{\infty} dx\, R(x) = \oint_C dz\, R(z) = 2\pi i \underbrace{\sum_{y>0}}_{\text{upper half plane}} \text{Res}\left[R(z)\right]} \quad (6)$$

<u>Example:</u>

$$I = \int_{-\infty}^{\infty} dx \frac{1}{(x^2+1)^2} = \oint_C dz \frac{1}{(z^2+1)^2} = \oint dz \frac{1}{[(z+i)(z-i)]^2} \quad (7)$$

C ← large Semi-circle

$$= \oint dz \frac{1}{(z+i)^2(z-i)^2}$$

← Only the root at $z=+i$ contributes, since the root at $z=-i$ is not inside the contour

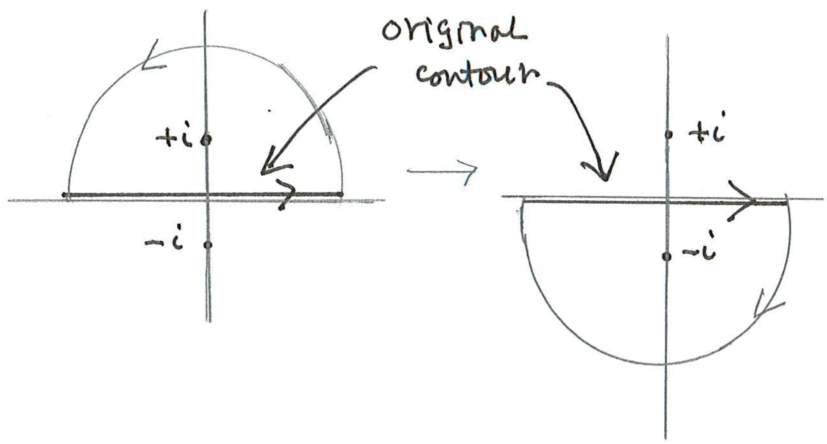
$$\text{Hence } I = \oint dz \frac{\left(1/(z+i)^2\right)}{(z-i)^2} = \frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{1}{z+i}\right)^2 \Bigg|_{z=i} = -\frac{4\pi i}{(z+i)^3}\Bigg|_{z=i} \quad (8)$$

$$\therefore \boxed{I = -\frac{4\pi i}{(2i)^3} = \frac{\pi}{2} = \text{REAL}} \quad (9)$$

COMMENTS:

(a) As a rough rule of thumb, if a real integral gives a result proportional to π, then it can be readily evaluated using contour integration, from which factors of 2π naturally arise.

(b) The original real integral was $\int_{-\infty}^{\infty} dx \cdots$, and we chose to form this into a closed contour by adding a semi-circle in the upper-half-plane (u.h.p). The natural question is what would have permitted us to arrive at the same result, had we closed the contour in the lower-half-plane?



$$I = \oint dz \frac{1}{(z+i)^2 (z-i)^2}$$ $\leftarrow$ now only the root at $z=-i$ contributes since $z=+i$ is outside the contour (10)

Hence $$I = \oint dz \frac{\left(1/(z-i)^2\right)}{(z+i)^2} = -\frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{1}{z-i}\right)^2\Bigg|_{z=-i} = +\frac{4\pi i}{(z-i)^3}\Bigg|_{z=-i} \qquad (11)$$

CW Contour!

$$= \frac{+4\pi i}{(-2i)^3} = \frac{\pi}{2} \qquad (12)$$

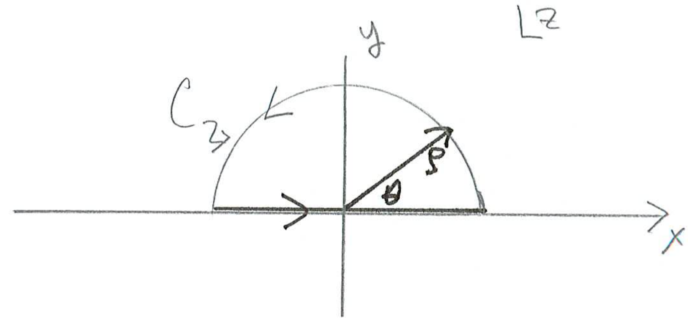
SAME AS BEFORE! (See (9))

CONCLUSION: $\int_{-\infty}^{\infty}$ can be made into a closed contour by adding a semi-circle in either the u.h.p or the l.h.p. EITHER WAY $\Rightarrow$ SAME RESULT!

(c) The next class of integrals we consider is

$$I=\int_{-\infty}^{\infty} dx\, R(x)\, e^{ix} = \int_{-\infty}^{\infty} dx\, R(x) \cos x + i \int_{-\infty}^{\infty} dx\, R(x) \sin x \qquad (1)$$

where $R(x)$ is a polynomial in x as before. The condition for the existence of these integrals is now that $|R(z)| \to 0$ uniformly in θ as $\rho \to \infty$. [JORDAN'S LEMMA]



$\rho \to \infty$ is understood

Following the previous discussion we can write

$$\boxed{I=\int_{-\infty}^{\infty} dx\, R(x) e^{ix} = \oint_C dz\, R(z)\, e^{iz} = 2\pi i \sum_{y>0} \mathrm{Res}\left[R(z)\, e^{iz}\right]} \qquad (2)$$

<u>Comments:</u> For this class of integrals the contour <u>must</u> be closed in the u.h.p. only (not the l.h.p) to ensure that in the vicinity of the imaginary axis the integrand is a damped exponential:

$$e^{iz} \xrightarrow{u.h.p.} e^{i(x+iy)} = e^{ix} e^{-y} \xrightarrow{\rho \to \infty} 0 \qquad (3)$$

Had the contour been closed in the l.h.p. we would have a diverging factor in the integrand:

$$e^{iz} \xrightarrow{l.h.p} \cancel{e^{ix}}\, e^{i(-iy)} = e^{ix} e^{+y} \xrightarrow{\rho \to \infty} \infty \qquad (4)$$

The situation is reversed if the integrand contains the factor e^{-iz}.

Example: $I = \int_{-\infty}^{\infty} dk \, \dfrac{e^{ikr}}{k^2+\mu^2}$ $\quad r = \text{fixed} > 0$ (5)

Let $x = kr \Rightarrow I = r\int_{-\infty}^{\infty} dx \, \dfrac{e^{ix}}{x^2+\Lambda^2}$ $\quad \Lambda = \mu r$ (6)

Then from 93(2): $I = r\oint_C dz \, \dfrac{e^{iz}}{z^2+\Lambda^2} = r \cdot 2\pi i \sum_{\text{Res } y>0} \left[\dfrac{e^{iz}}{z^2+\Lambda^2}\right]$ (7)

The poles are determined by $\dfrac{e^{iz}}{z^2+\Lambda^2} = \dfrac{e^{iz}}{(z+i\Lambda)(z-i\Lambda)} = \dfrac{e^{iz}/(z+i\Lambda)}{z-i\Lambda}$ (8)

↑ this is in u.h.p.

Since the only pole in the u.h.p. is $z = +i\Lambda$ we find:

$$I = r \cdot 2\pi i \left[\frac{e^{iz}}{z+i\Lambda}\right]_{z=i\Lambda} = 2\pi i r\left[\frac{e^{i(i\Lambda)}}{2i\Lambda}\right] = \frac{\pi r}{\Lambda} e^{-\Lambda} \quad (9)$$

$$\therefore \boxed{I = \frac{\pi r}{\Lambda} e^{-\Lambda} = \pi \frac{e^{-\mu r}}{\mu}} \quad (10)$$

Comments: When given an integral of the form $I_1 = \int_{-\infty}^{\infty} dx \, \dfrac{\cos x}{x^2+a^2}$ (11)

We can evaluate it in 2-ways:

(a) $I_1 = \text{Re}\oint_C dz \, \dfrac{e^{iz}}{z^2+a^2} = \text{Re}\left[\pi \dfrac{e^{-a}}{a}\right] = \pi \dfrac{e^{-a}}{a}$ using (7) & (10) (12)

Note incidentally that $I_2 = \int_{-\infty}^{\infty} dx \, \dfrac{\sin x}{x^2+a^2} = \text{Im}\oint_C dz \, \dfrac{e^{iz}}{z^2+a^2}$ (13)

$= \text{Im}\left[\pi \dfrac{e^{-a}}{a}\right] = 0$ (14)

This makes sense, since I_2 is the integral of an odd function over a symmetric interval.

(b) The other way of writing the integral is:

$$\cos x = \frac{1}{2}(e^{ix}+e^{-ix}) \Rightarrow I = \frac{1}{2}\oint dz\,(e^{iz}+e^{-iz})\frac{1}{z^2+a^2} \qquad (15)$$

However, now we have to treat e^{iz} and e^{-iz} separately to ensure convergence:

$$I = \underbrace{\frac{1}{2}\oint_C dz\,\frac{e^{iz}}{(z+ia)(z-ia)}}_{I_3 = \text{close in u.h.p}} + \underbrace{\frac{1}{2}\oint_C dz\,\frac{e^{-iz}}{(z+ia)(z-ia)}}_{I_4 = \text{close in l.h.p.}} \qquad (16)$$

$$I_3 = \frac{1}{2} I_1 = \frac{1}{2}\pi\frac{e^{-a}}{a} \qquad (17)$$

CW contour!

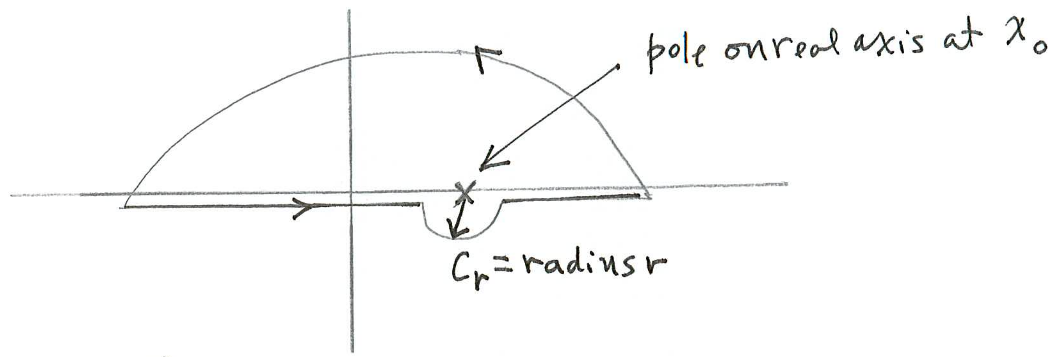
$$I_4 = \frac{1}{2}\oint_{\substack{C \\ \text{l.h.p.}}} dz\,\frac{e^{-iz}/(z-ia)}{z+ia} = -2\pi i\frac{1}{2}\left[\frac{e^{-iz}}{z-ia}\right]_{z=-ia} \qquad (18)$$

$$I_4 = -\pi i\,\frac{e^{-i(-ia)}}{-2ia} = \frac{\pi}{2}\frac{e^{-a}}{a} \qquad (19)$$

$$\therefore \boxed{I = I_3 + I_4 = \frac{1}{2}\pi\frac{e^{-a}}{a} + \frac{1}{2}\pi\frac{e^{-a}}{a} = \pi\frac{e^{-a}}{a}} \qquad (20)$$

This is the same result obtained in (12) using the previous method.

CONTOUR INTEGRALS INVOLVING POLES ON THE REAL AXIS



Consider $I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x - x_0} \longrightarrow \oint_C dz\, \frac{f(z)}{z - x_0}$ (1)

Principal value

$$\oint_C dz \ldots = \int_{-\infty}^{x_0 - r} + \int_{x_0 + r}^{\infty} + \oint_{C_r} (\text{small semicircle}) + \int (\text{large semicircle}) \quad (2)$$

$\rightarrow 0$ — assume this to hold

$\rightarrow 2\pi i \left[\text{Residue of pole at } x_0\right] = 2\pi i f(x_0)$ (3)

Hence $\underbrace{\int_{-\infty}^{x_0 - r} \ldots + \int_{x_0 + r}^{+\infty} \ldots}_{\equiv P\int_{-\infty}^{\infty} dx \ldots} = 2\pi i f(x_0) - \oint_{C_r} dz\, \frac{f(z)}{z - x_0}$ (4)

As the radius r of the small semi-circle decreases the l.h.s. of (4) becomes the integral I that we are trying to evaluate. Hence

$$I = 2\pi i f(x_0) - \oint_{C_r} dz\, \frac{f(z)}{z - x_0} \overset{r \to 0}{=} 2\pi i f(x_0) - f(x_0) \oint dz\, \frac{1}{z - x_0} \quad (5)$$

along semicircle

Along the semi-circle: $z - x_0 = re^{i\theta}$; $dz = ire^{i\theta} d\theta$

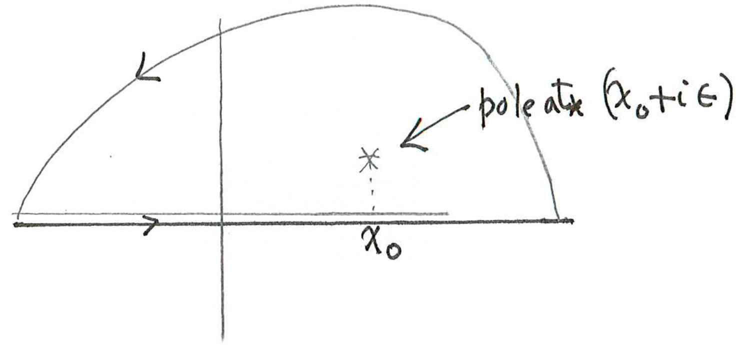
$$\oint dz \ldots \longrightarrow \int_{-\pi}^{0} d\theta \cdot \frac{ire^{i\theta}}{re^{i\theta}} = i\int_{-\pi}^{0} d\theta = +\pi i \quad (6)$$

Collecting together the results in (5) & (6) we have:

$$\boxed{I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0} = 2\pi i f(x_0) - \pi i f(x_0) = \pi i f(x_0)} \quad (7)$$

We can infer from this a "rule":

A pole of the form $\frac{f(z)}{z-z_0}$ in the complex plane (but not on the real axis) contributes a reside $\underline{2\pi i f(z_0)}$ at the pole. <u>However, if the pole lies on the real axis its contribution is $\pi i f(x_0)$.</u>



Check on Result:

The result in Eq. (7) above can be checked by displacing the pole as shown by an amount $+i\epsilon$, and then taking the limit $\epsilon \to 0$. Again we wish to evaluate

$$I = P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0} = \lim_{\epsilon\to 0} \underbrace{\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-(x_0+i\epsilon)}}_{} \quad (8)$$

$$\to \oint dz\, \frac{f(z)}{z-(x_0+i\epsilon)} = 2\pi i f(x_0+i\epsilon)$$

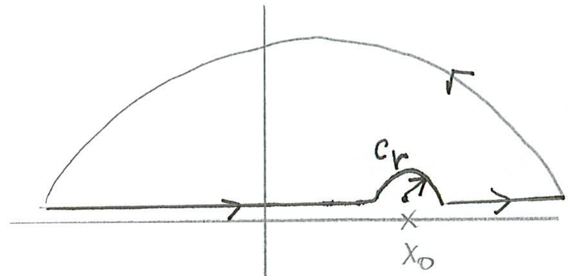
$$\to 2\pi i f(x_0) \quad (9)$$

DIRAC IDENTITY

$$\therefore 2\pi i f(x_0) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\, f(x)\left\{\frac{1}{x-x_0-i\epsilon}\right\} = \underbrace{P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-x_0}}_{I} + \underbrace{i\pi\int_{-\infty}^{\infty} dx\, \delta(x-x_0)\, f(x)}_{i\pi f(x_0)} \quad (10)$$

Hence $$\boxed{I = 2\pi i f(x_0) - i\pi f(x_0) = i\pi f(x_0) \quad \checkmark} \quad (11)$$

We note that the same result can be obtained by deforming the original contour so as to exclude the pole on the real axis:



entire contour $\equiv C$

As before: $$\oint_C dz \ldots = \underbrace{\int_{-\infty}^{x_0-r} + \int_{x_0+r}^{\infty}}_{P\int_{-\infty}^{\infty} dx \ldots} + \int_C (\text{small semicircle}) + \cancel{\int (\text{large semi-circle})}^{\,0} \qquad (12)$$

NOTE CHANGE!

$$= 2\pi i \left[\text{Residues inside } C\right] \equiv 0 \qquad (13)$$

Hence from (12) & (13): $$I = P\int_{-\infty}^{\infty} dx \, \frac{f(x)}{x-x_0} = -\oint_{C_r} \qquad (14)$$

Following the discusson on p.97 we have:

$$-\oint_{C_r} = -f(x_0)\int_{\pi}^{0} i\, d\theta = -f(x_0)\, i\,(0-\pi) = +i\pi f(x_0) \qquad (15)$$

Hence from (14) & (15): $$\boxed{I = P\int_{-\infty}^{\infty} dx \, \frac{f(x)}{x-x_0} = +i\pi f(x_0)} \qquad (16)$$

This agrees with the results obtained previously in Eqs. (7) and (11).

Conclusion: There is often more than one way to evaluate a contour integral containing poles. We have seen 3 methods here:
a) Include the pole in the closed contour. b) Exclude the pole from the closed contour. c) Displace the pole off the real axis.

APPLICATION Evaluate $I_1 = P\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$

$$I_2 = P\int_{-\infty}^{\infty} dx \, \frac{\cos x}{x} \tag{1}$$

Using the previous results we first note that

$$I_1 = \mathrm{Im}\left\{P\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x}\right\} \; ; \; I_2 = \mathrm{Re}\left\{P\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x}\right\} \tag{2}$$

Hence we evaluate $P\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x} = \pi i \left[\text{Residue of pole at } x=0\right]$ (3)

$$= \pi i \left[e^{iz}\right]_{z=x=0} = \pi i \cdot 1$$

$$\therefore I_1 = \mathrm{Im}[i\pi] = \pi \Rightarrow \boxed{P\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} = \pi} \tag{4}$$

$$I_2 = \mathrm{Re}[i\pi] = 0 \Rightarrow \boxed{P\int_{-\infty}^{\infty} dx \, \frac{\cos x}{x} = 0} \tag{5}$$

COMMENTS:

1) The result in (4) is in agreement with the result previously obtained on p. CV-54(3) using Hilbert transform pairs.

2) The result in (5) follows by symmetry: The integrand is an odd function of x being integrated over a symmetric interval.

3) Despite appearances the integrand in (4) does not actually have a pole at $x=0$: Recall that $\sin x/x \to 1$ as $x \to 0$. Nonetheless this formalism applies.

4) When the contours in (1) and (2) are formed into closed contours in the u.h.p., the integral over the large semi-circle $\to 0$ by the usual arguments: $e^{iz} \underset{\text{u.h.p}}{\to} e^{i(x+iy)} \to e^{-y} e^{ix} \to 0$.