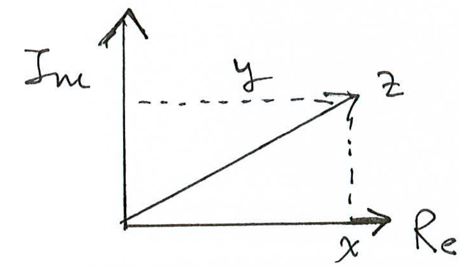
COMPLEX VARIABLES

Let $i = \sqrt{-1}$; $z = x + iy$ $\quad$ (x, y are real) $\quad$ (1)

Pictorially:



The theory of complex numbers can be developed by viewing them as 2-dim vectors. From (1) & figure we develop the following simple rules:

<u>addition</u> : $z_1 = a + ib \qquad z_2 = c + id \Big\} \Rightarrow (z_1 + z_2) = (a+c) + i(b+d)$ $\quad$ (2)

<u>multiplication</u> : $z_1 z_2 = (a+ib)(c+id) = ac + i^2 bd + i(bc + ad)$

$$\boxed{\text{New Vector} = z_1 z_2 = \underbrace{(ac - bd)}_{x\text{-component}} + i \underbrace{(bc + ad)}_{y\text{-component}}} \quad (3)$$

Recall ; $i^2 = -1$; $i^3 = -i$; $i^4 = +1$ $\quad$ (4)

In a practical sense the presence of $i = \sqrt{-1}$ merely serves to define a prescription for multiplication, which can be summarized compactly as

$$\boxed{(a, b)(c, d) = (ac - bd, \ bc + ad)} \quad (5)$$

The theory of complex numbers can be developed using (5) directly instead of (3).

<u>division</u>: $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2}\left(\dfrac{x_2 - iy_2}{x_2 - iy_2}\right) = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \dfrac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$ $\quad$ (6)

<u>Complex Conjugation</u> : $\boxed{z = x + iy \Rightarrow z^* = \bar{z} \equiv x - iy}$ $\quad$ (7)

More generally: $i \to -i$

Rules for Complex Conjugation

1) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$
2) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$
3) $z+\bar{z} = 2x = 2\,\mathrm{Re}\, z$
4) $z-\bar{z} = 2iy = 2i\,\mathrm{Im}\, z$

Proofs are trivial: For example: $\overline{z_1 z_2} = (ac-bd) - i\,(bc+ad)$

Compare to $\bar{z}_1 \bar{z}_2 = (a-ib)(c-id) = (ac-bd) - i\,(bc+ad)$

Absolute Value = Modulus = Magnitude of a Complex Number :

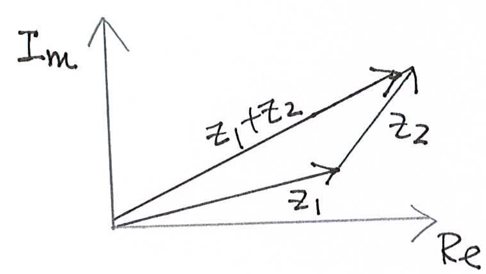
$$|z| \equiv |x+iy| \equiv \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

Evidently: ① $|z|^2 = x^2+y^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$

③ $|z| \geq \mathrm{Re}(z)$; $|z| \geq \mathrm{Im}(z)$

③ $|z_1 z_2| = |z_1||z_2|$

④ $|z_1+z_2| \leq |z_1| + |z_2|$ [triangle inequality]



Complex Numbers in Polar Coordinates:

Depending on the problem polar coordinates may be more useful than Cartesian coordinates:

$x \to r\cos\theta$; $y \to r\sin\theta$; $z = x+iy \to r(\cos\theta + i\sin\theta)$

$= r\,e^{i\theta}$

$\hookrightarrow \sqrt{x^2+y^2}$

Hence: $\boxed{r = \sqrt{x^2+y^2} = |z| \;;\; \theta = \tan^{-1}(y/x) \equiv \arg z}$

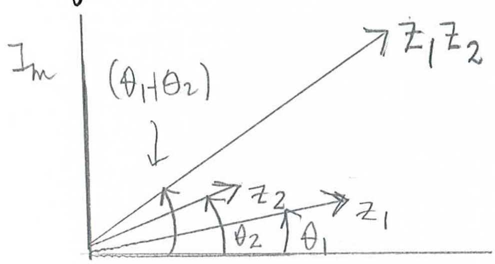
Key Problem with Polar Coordinates: θ is not unique

$\Rightarrow$ branch cuts, ... (more later!!)

Notes: ① $r = e^{i\theta} = e^{i(\theta+2\pi)} = \dots = e^{i(\theta+2n\pi)}$ $\quad n = 0, 1, 2, \dots$

② multiplication: $z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$ (1)

Physical Interpretation: Multiplication of one complex number by another changes the length of the number being multiplied and also rotates it:



$\boxed{\text{De Moivre's Theorem:}}$ $z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$ (2)

$\therefore \boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$ (3)

n can be an integer or any rational number here

Application: Find the nth root(s) of a given complex number z:

(Find z_0 such that $z^{1/n} = z_0$) $\Rightarrow (z^{1/n})^n = z_0^n \Rightarrow z_0^n = z$ (4)

Write $z_0 = r_0 e^{i\theta_0}$; $z = re^{i\theta} \Rightarrow r_0^n e^{in\theta_0} = re^{i\theta}$

Principle: When equating two complex numbers in Cartesian space write

$\left.\begin{array}{l} z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{array}\right\} \Rightarrow$ If $z_1 = z_2$ then: $\begin{array}{l} x_1 = x_2 \\ y_1 = y_2 \end{array}$

In polar coordinates: $\left.\begin{array}{l} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{array}\right\}$ If $z_1 = z_2$ then $r_1 = r_2$ (5)

but $\theta_1 = \theta_2 \pm 2n\pi$

$\hookrightarrow$ leads to multivalued functions

Application: $z^{1/n} = z_0 \Rightarrow r e^{i\theta} = r_0^n e^{in\theta_0}$

$\therefore\ r = r_0^n \Rightarrow \boxed{r_0 = r^{1/n}}$ (6) $\quad \theta = n\theta_0 \pm \underbrace{2\pi k}_{e^{i2\pi k}=1}$ ← integer

$$\therefore \boxed{\theta_0 = \frac{\theta}{n} \pm \frac{2\pi k}{n}} \quad (7)$$

Hence the full solution z_0 is given by $z_0 = r_0 e^{i\theta_0} \Rightarrow$

$$z_0 = r^{1/n} e^{i\left(\frac{\theta}{n} \pm 2\pi\frac{k}{n}\right)} \quad (8)$$

$$\boxed{z_0 = r^{1/n} e^{i\theta/n} e^{2\pi i\left(\frac{n-k}{n}\right)}} \quad (9)$$

Q: Since k is an arbitrary integer how many distinct roots do we find

A: n roots

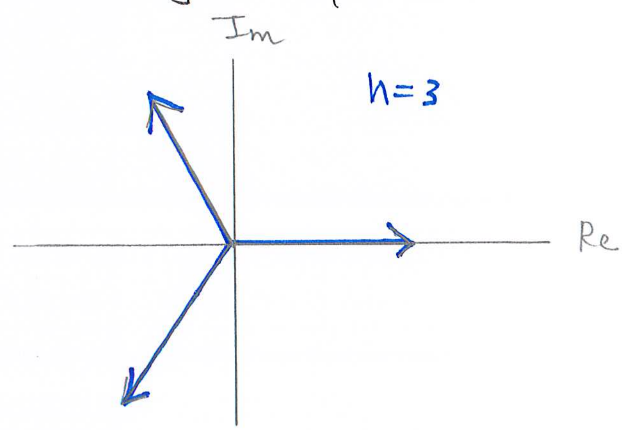
Example: $n=3$: $e^{2\pi i\left(\frac{n-k}{n}\right)} = \underbrace{e^{2\pi i}}_{1}$ $(k=0)$; $e^{2\pi i \cdot \frac{2}{3}}$ $(k=1)$; $e^{2\pi i(\frac{1}{3})}$ $(k=2)$

$\underbrace{e^{2\pi i \cdot 0}}_{1}$ $(k=3)$; $e^{2\pi i(-\frac{1}{3})}$ $(k=4) = \underbrace{e^{2\pi i}e^{-\frac{2\pi i}{3}}}_{e^{4\pi i/3}}$ $(k=4)$

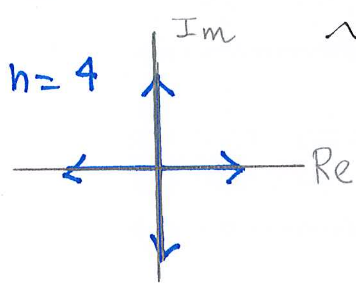
Hence after a while the roots repeat leaving only 3 independent roots:

$1, \quad e^{i\frac{4}{3}\pi}, \quad e^{i\frac{2}{3}\pi}$

$0°$ $\quad 240°$ $\quad 120°$



$r = 1 \Rightarrow$ "nth roots of unity"



Analytic Functions: Cauchy-Riemann Conditions

Any function $w = f(z)$ can be written in the form

$$w(z) = u(x,y) + i\,v(x,y)$$

Ex: $$w = f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)}_{u(x,y)} + \underbrace{2ixy}_{iv(x,y)} \qquad (1)$$

It is critical to identify those functions which have derivatives. Such functions are said to be analytic

$$\boxed{\text{analytic} \longleftrightarrow \text{differentiable (a \underline{unique} derivative exists)}} \qquad (2)$$

Some functions may be analytic everywhere in the complex plane except at isolated points ("poles") or lines ("branch cuts") ....

Consider ① $$w = f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y) \qquad (3)$$

$$= \underbrace{e^x \cos y}_{u(x,y)} + i\,\underbrace{e^x \sin y}_{v(x,y)} \qquad (4)$$

$\longleftarrow$ this function is <u>analytic everywhere</u>

② $$w = f(z) = \bar{z} = x - iy \quad ; \quad u(x,y) = x \qquad v(x,y) = -y \qquad (5)$$

This function is <u>not analytic</u>

Cauchy-Riemann Conditions:

We will show shortly that there is a simple test for analyticity:

$w(z) = u(x,y) + i\,v(x,y)$ is analytic iff

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \qquad (6)$$

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__Details__: $f(z)$ is analytic at z_0 if the following limit exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \qquad (1)$$

"Exists" $\Rightarrow$ SAME limit however $z \to z_0$

__Notation__: Analytic = differentiable = regular = holomorphic

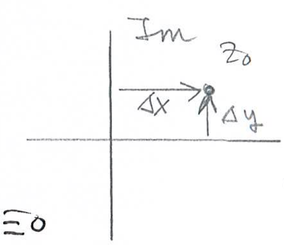
__Examples__: Start with $f'(z_0) = \lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ (2)

(a) Consider $f(z) = z^2 \Rightarrow f'(z_0) = \dfrac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \cong \dfrac{\cancel{z_0^2} + 2 z_0 \Delta z - \cancel{z_0^2}}{\Delta z}$ (3)

$$\boxed{\therefore f'(z_0) = \lim_{\Delta z \to 0} \frac{2 z_0 \cancel{\Delta z}}{\cancel{\Delta z}} = 2 z_0 \text{ (independent of } \Delta z\text{!)}} \qquad (4)$$

(b) Next consider $f(z) = \bar{z} \Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \dfrac{\overline{(z_0 + \Delta z)} - \bar{z}_0}{\Delta z}$ (5)

$$\therefore f'(z_0) = \lim_{\Delta z \to 0} \left(\frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i \Delta y}{\Delta x + i \Delta y} \right) \qquad (6)$$



If the limit is taken in the x-direction then $\Delta y \equiv 0$

and $f'(z_0) = \lim_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1 \longleftarrow$ (7a)

However, if the limit is taken in the y-direction then $\Delta x = 0$ and

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{-i \Delta y}{i \Delta y} = -1 \longleftarrow \qquad (7b)$$

Since the limit depends on the path, $f(z) = \bar{z}$ is not analytic.

Derivation of Cauchy-Riemann Conditions:

The preceding examples of path-independence (or not!) lead to the formal proof of the C-R conditions:

(a) First assume that $W(z)$ is analytic; Then we show neccesity of C-R:

$$W'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta u + i\Delta v}{\Delta z} \right) = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z} \right) \quad (1)$$

Multiply numerators & denominators by $\frac{\Delta x - i\Delta y}{\Delta x - i\Delta y}$

$$W'(z_0) = \lim_{\Delta z \to 0} \left[\frac{\Delta u(\Delta x - i\Delta y)}{(\Delta x)^2 + (\Delta y)^2} + i\frac{\Delta v(\Delta x - i\Delta y)}{(\Delta x)^2 + (\Delta y)^2} \right] \quad (2)$$

Collecting real & imaginary terms gives

$$W'(z_0) = \lim_{\Delta z \to 0} \left[\frac{\Delta u \Delta x + \Delta v \Delta y}{(\Delta x)^2 + (\Delta y)^2} + i\frac{\Delta v \Delta x - \Delta u \Delta y}{(\Delta x)^2 + (\Delta y)^2} \right] \quad (3)$$

Since $W(z)$ is assumed to be analytic we must obtain the same derivative independent of how $\Delta z \to 0$ is taken. Take $\underline{\Delta y = 0}$ initially; Then

$$W'(z_0) = \lim_{\Delta x \to 0} \left[\frac{\Delta u \Delta x}{(\Delta x)^2} + i\frac{\Delta v \Delta x}{(\Delta x)^2} \right] = \lim_{\Delta x \to 0} \underbrace{\left[\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \right]} \quad \text{when } \Delta y \equiv 0 \quad (4)$$

Next take $\underline{\Delta x = 0}$ so that $\Delta z = i\Delta y \Rightarrow$

$$W'(z_0) = \lim_{i\Delta y \to 0} \left[\frac{\Delta v \Delta y}{(\Delta y)^2} - i\frac{\Delta u \Delta y}{(\Delta y)^2} \right] = \lim_{i\Delta y \to 0} \underbrace{\left[\frac{\Delta v}{\Delta y} - i\frac{\Delta u}{\Delta y} \right]} \quad \text{when } \Delta x \equiv 0 \quad (5)$$

Since $W(z)$ is analytic the expressions ~~~ in (4), (5) must be equal. Equating real and imaginary parts, and going to the limit gives:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad \text{C-R CONDITIONS} \tag{6}$$

Hence if $W(z)$ is analytic then the C-R Conditions hold.

(b) Next we prove the converse: If the C-R conditions hold then $W(z)$ is analytic: (Sufficiency of C-R) [f(z) is assumed continuous]

$$f(z) = u(x,y) + iv(x,y) \Rightarrow \Delta f = f(z_0+\Delta z) - f(z_0) = \Delta u + i\Delta v \tag{7}$$

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \text{ where } \epsilon_{1,2} \to 0 \text{ as } \Delta x, \Delta y \to 0$$

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y, \quad \text{"} \quad \epsilon_{3,4} \to 0 \cdots \tag{8}$$

$-\partial v/\partial x$ ← C-R

$$\therefore \Delta f = \Delta u + i\Delta v = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2 \Delta y \tag{9}$$

$$+ i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y\right)$$

$\partial u/\partial x$ ← C-R (10)

$$\text{Hence: } \Delta f = \frac{\partial u}{\partial x}\Delta x + \left(-\frac{\partial v}{\partial x}\Delta y\right) + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y\right] + \text{terms} \to 0 \tag{11}$$

$$= \frac{\partial u}{\partial x}\underbrace{(\Delta x + i\Delta y)}_{\Delta z} + i\frac{\partial v}{\partial x}\underbrace{(\Delta x + i\Delta y)}_{\Delta z} \tag{12}$$

$$\text{Dividing by } \Delta z \Rightarrow \boxed{\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \underline{\text{INDEPENDENT OF } \Delta z}} \tag{13}$$

This establishes that the C-R Conditions are sufficient to ensure the analyticity of $f(z)$: the fact that the derivative is independent of path.

Examples: ① $f(z) = e^z = e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$ (14)

$$\frac{\partial u}{\partial x} = e^x \cos y \overset{?}{=} \frac{\partial v}{\partial y} = e^x \cos y \ \checkmark \qquad (15)$$

$$\frac{\partial v}{\partial x} = e^x \sin y \overset{?}{=} -\frac{\partial u}{\partial y} = -e^x(-\sin y) \ \checkmark \qquad (16)$$

Note that for $f(z) = e^z$ the C-R conditions hold everywhere as an identity: Such a function is said to be "entire".

Since $f(z) = e^z$ is analytic everywhere its derivative can be computed along any path:

(a) $f(z) = e^x \cos y + i e^x \sin y$ (7)

$$\Delta z = \Delta x \Rightarrow \quad \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y)$$

$$= e^x e^{iy} = e^{x+iy} = e^z \ \checkmark \qquad (18)$$

(b) $f(z) = e^x \cos y + i e^x \sin y$

$$\Delta z = i\Delta y \Rightarrow \quad \frac{df}{dz} = \frac{\partial u}{i\partial y} + i\frac{\partial v}{i\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = e^x(\cos y) - i e^x(-)\sin y)$$

$$= e^x(\cos y + i \sin y) = e^z \ \checkmark \qquad (19)$$

(c) $f(z) = e^z \qquad \dfrac{df}{dz} = e^z \ \checkmark$ (20)

$\hookrightarrow$ any path $\Delta z \to 0$

Note that Eqs. (18), (19), (20) give the same result!

Examples (continued)

Consider next $f(z) = |z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2+y^2$ (21)

$$= u(x,y) + i\,v(x,y) \Rightarrow u(x,y) = x^2+y^2 \,;\; v(x,y) \equiv 0$$

$$\frac{\partial u}{\partial x} = 2x \Leftrightarrow \frac{\partial v}{\partial y} = 0 \quad ; \quad \frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y} = -2y \tag{22}$$

Hence the C-R conditions hold only at the origin $(x=y=0)$; [We would not call a function analytic if C-R hold only at 1 point.]

Note for later: Since $z = x+iy$ and $\bar{z} = x-iy$ we have

$$\boxed{x = \frac{1}{2}(z+\bar{z}) \quad ; \quad y = \frac{1}{2i}(z-\bar{z})} \tag{23}$$

Hence any function $f = u(x,y) + i\,v(x,y) \to f(z,\bar{z})$. We will later show that any function $f = f(x,y)$ which depends on $\bar{z}$ (in addition to z) when use is made of (23) is not analytic. $f(z) = |z|^2 = z\bar{z}$ is an example.

General Rules on Analytic Functions

a) a constant is analytic

b) z^n is analytic

c) the sum, or product of 2 analytic functions is analytic

d) the quotient of 2 analytic functions is analytic, provided that the denominator $\neq 0$

c) an analytic function of an analytic function is analytic (CHAIN RULE);

Example:

$f(z) = z^2 \qquad g(z) = e^z \Rightarrow g(f(z)) = e^{z^2} = \text{analytic}$

Side Comment: Consider $f = u + iv \overset{C-R}{\Rightarrow} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Compare to $\underset{f'}{\underset{\shortparallel}{if}} = iu + i(iv) = \underbrace{-v}_{u'} + \underbrace{iu}_{v'}$

for f' C-R $\Rightarrow \frac{\partial u'}{\partial x} = \frac{\partial v'}{\partial y}; \frac{\partial v'}{\partial x} = -\frac{\partial u'}{\partial y}$

$$-\frac{\partial v}{\partial x} \overset{?}{=} \frac{\partial u}{\partial y} \checkmark; \quad \frac{\partial u}{\partial x} = -\frac{(-\partial v)}{\partial y} = \frac{\partial v}{\partial y} \checkmark$$

Hence if f is analytic the function $i \cdot f$ is also analytic

Since the factor of i interchanges u and v with the right phases.

Connection To Physics: Harmonic Functions

$$\text{CR} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (1)$$

$$\Downarrow$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \; ; \; \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

Hence $$\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \equiv 0 \qquad (3)$$

$$\boxed{\nabla^2 u = 0 \quad \text{also } \nabla^2 v = 0} \qquad (4)$$

$u(x,y)$ and $v(x,y)$ are <u>harmonic functions</u>. If $f(z) = u + iv$ is analytic then $u(x,y)$ and $v(x,y)$ are harmonic, and are called <u>Conjugate harmonic functions</u>. Given $u(x,y)$ or $v(x,y)$ we can find the other one using the C-R conditions:

<u>Ex:</u> (a) Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic
(b) find $v(x,y)$ its harmonic conjugate

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2 - 3x^2 + 3y^2 \; ; \; \frac{\partial^2 u}{\partial x^2} = -6x \\ \frac{\partial u}{\partial y} &= 6xy \; ; \; \frac{\partial^2 u}{\partial y^2} = +6x \end{aligned}\right\} \Rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0 \checkmark \qquad (5)$$

$\Rightarrow u(x,y)$ is harmonic

To find $v(x,y)$: $\dfrac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \overset{\text{C-R}}{=} \dfrac{\partial v}{\partial y} \Rightarrow v(x,y) = 2y - 3x^2y + y^3 + \mathcal{Y}(x)$

$$\Downarrow$$

To fix $\mathcal{Y}(x)$ use the other C-R relation; $\partial v/\partial x = -\partial u/\partial y$; $\partial v/\partial x = -6xy + \mathcal{Y}'(x)$ (6)

But $\partial v/\partial x = -\partial u/\partial y = -6xy \Rightarrow \mathcal{Y}'(x) = 0 \Rightarrow \mathcal{Y}(x) = \text{const}$

Hence $$\boxed{v(x,y) = 2y - 3x^2y + y^3 + \text{const}} \qquad (7)$$

We can use this result to illustrate an important theorem: $\boxed{\text{If } W(z) = u(x,y) + i\,v(x,y) \text{ is analytic iff } \frac{dW}{d\bar{z}} \equiv 0}$

Note: When we use the notation $f(z)$ or $W(z)$ for an function of a complex variable, our notation is a bit sloppy: As already noted, any function $f = u(x,y) + i\,v(x,y)$ can be expressed in terms of z AND $\bar{z}$ using Eq. (23) p. 13:

$$\boxed{x = \frac{1}{2}(z+\bar{z}) \;;\; y = \frac{1}{2i}(z-\bar{z})} \qquad (8)$$

When we write $f(z)$ we are not necessarily saying that f does not also depend on $\bar{z}$. However, what the theorem says is that if f (or W) is analytic, then in fact it does not depend on $\bar{z}$, but only on z.

Returning to the previous example we have

$$f = u(x,y) + i\,V(x,y) = \left[2x - x^3 + 3xy^2\right] + i\left[2y - 3x^2y + y^3 + \text{const}\right] \qquad (9)$$

Substituting for x & y using (8) above we find

$$\boxed{f(x,y) \rightarrow f(z,\bar{z}) = 2z - z^3 + C} \qquad (10)$$

Hence, even though f could have depended on $\bar{z}$ as well as on z, in fact it only depends on z. This is what the theorem tells us! We know that f must be analytic because $u(x,y)$ and $V(x,y)$ are harmonic conjugates of each other. This theorem then says that when f is analytic then $f = f(z)$ only.

Proof! $$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} \qquad (11)$$

$$(8) \Rightarrow \frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \;;\; \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$$

Hence $\dfrac{\partial f}{\partial \bar{z}} = \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)\dfrac{1}{2} + \left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)\left(\dfrac{-1}{2i}\right)$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\substack{\parallel \\ 0}} + \frac{i}{2}\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{\substack{\parallel \\ 0}} = 0 \qquad \text{Q.E.D}$$

$$\leftarrow \text{C-R} \longrightarrow$$

Returning to the previous example we could have guessed the form of $f(z)$ by noting that along the real axis where $y = 0$ we have

$$f = u(x,y) + iv(x,y) = [2x - x^3 + 3xy^2] + i[2y - 3x^2y + y^3 + \text{const}]$$

$$\longrightarrow [2x - x^3] + i[0] + \text{const}$$

This is the same expression as would have been obtained from $f(z) = 2z - z^3 + c$ along the real axis, which gives the previous answer.

2-DIMENSIONAL ELECTROSTATICS

Some electrostatics problems have a 2-dimensional geometry (with a symmetry in the 3rd dimension), that lend themselves to the use of complex variables.

Consider a charge-free region of space with some conductors. The electrostatic potential $\varphi(\vec{x})$ is constant on these surfaces [since otherwise we would have $\vec{E} = -\vec{\nabla}\varphi \neq 0 \Rightarrow$ flow of charge in a static situation].

Then

$$\vec{E} = -\vec{\nabla}\varphi \quad ; \quad \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \nabla^2\varphi = 0 \qquad (1)$$

Write

$$0 = \vec{\nabla}\cdot\vec{E} = \partial E_x/\partial x + \partial E_y/\partial y \Rightarrow \frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y} \qquad (2)$$

Additionally: Since $\vec{\nabla}\cdot\vec{E} = 0$ in this circumstance $\vec{E}$ & $\vec{B}$ behave somewhat similarly so that we can write $\vec{E} = \vec{\nabla}\times\vec{A}$ where $\vec{A}$ is an appropriate potential.

This gives

$$E_x = \partial_y A_z - \partial_z A_y \quad ; \quad E_y = \partial_z A_x - \partial_x A_z \quad ; \quad E_z = \partial_x A_y - \partial_y A_x \qquad (3)$$

To generate a 2-dim field so that $E_z = 0 \Rightarrow A_y = A_x = \text{const} = 0$. Then $\vec{E}(x,y)$ and $\vec{A}(x,y)$ are given by

$$\underset{-\partial_x\varphi}{\overset{\shortparallel}{E_x}} = \partial_y A_z \Rightarrow -\frac{\partial\varphi}{\partial x} = \frac{\partial A}{\partial y} \quad ; \quad \text{Also } \underset{-\partial_y\varphi}{\overset{\shortparallel}{E_y}} = -\partial_x A_z \; ; \; \vec{A} = \hat{k}A = \hat{k}A_z \qquad (4)$$

So altogether:

$$\boxed{-\frac{\partial\varphi}{\partial x} = \frac{\partial A}{\partial y} \; ; \; \frac{\partial\varphi}{\partial y} = \frac{\partial A}{\partial x}} \qquad (5)$$

It follows that if we define an analytic function $f(z)$ such that

$$\boxed{f(z) = \varphi(x,y) - iA(x,y)} \qquad (6)$$

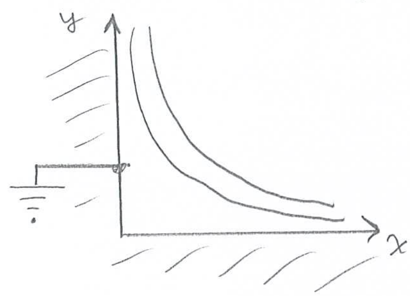
Then (5) are the C-R conditions for the complex potential.

Since $f(z)$ is analytic we can compute its derivative along any path:

$$\frac{df(z)}{dz} = \underbrace{\frac{\partial \Psi(x,y)}{\partial x}}_{-E_x} - i\underbrace{\frac{\partial A(x,y)}{\partial x}}_{\partial\Psi/\partial y = -E_y} = -E_x + iE_y = -(E_x - iE_y) = -\bar{E} \quad (7a)$$

Alternatively: $$\frac{df}{dz} = \frac{\partial \Psi}{i\partial y} - i\frac{\partial A}{i\partial y} = -i\underbrace{\frac{\partial \Psi}{\partial y}}_{-E_y} \underbrace{-\frac{\partial A}{\partial y}}_{-\frac{\partial \Psi}{\partial x} = E_x} = -(E_x - iE_y) = -\bar{E} \quad (7b)$$

Application: We show how the fact that $\Psi(x,y)$ is the real part of an analytic function can be utilized, by calculating the field of a grounded conductor formed into a right angle:



We want to find an analytic function $f(z) = \Psi(x,y) - iA(x,y)$ whose real part vanishes along $x = y = 0$.

Guess: $\Psi(x,y) = k\,xy$ $\qquad k =$ constant

We then guess that this is the real part of the analytic function

$$f(z) = \frac{-i}{2}kz^2 \quad (8)$$

Check: $f(z) = \frac{-i}{2}k(x+iy)(x+iy) = \frac{-i}{2}k\left[(x^2-y^2)+2ixy\right] = xy\cdot k - \frac{i}{2}k(x^2-y^2)$

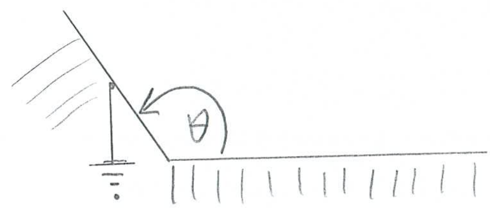
$$\therefore \boxed{f(z) = \frac{-i}{2}kz^2 = \underbrace{k\,xy}_{\Psi} \underbrace{-\frac{i}{2}k(x^2-y^2)}_{-iA}} \quad (9)$$

Then $E_x = -\frac{\partial \Psi}{\partial x} = ky$; $E_y = -\frac{\partial \Psi}{\partial y} = kx \qquad (10)$

CV- 16.2/16.3

Conformal Transformations: (Locally angle-preserving)

Having shown that this geometry can be described by an analytic function $f(z)$, we can replace z by some function of z which has the effect of mapping this geometry into another, e.g.,



A transformation which does this is called the <u>SCHWARZ TRANSFORMATION</u> and an example is

$$z' = z^{\beta}$$

For appropriate choice of β this maps a flat surface into one with an angle, as shown.

For more details see PANOFSKY & PHILLIPS, <u>Classical Electricity & Magnetism</u> pages 66-72.

Also: E. Durand, <u>Électrostatiques et Magnétostatiques</u>

Elementary Analytic Functions

We describe the properties of various functions that commonly arise

a) exponential: $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$ (1)

We have shown that C-R $\Rightarrow$ e^z is analytic everywhere. We have show also that $\frac{d}{dz}e^z = e^z$ as for real functions: Note...

$$\frac{d}{dz}e^z = \frac{\partial}{\partial x}\left[e^x(\cos y + i\sin y)\right] = e^x(\cos y + i\sin y) = e^z \checkmark \quad (2a)$$

$$\text{or}\quad \frac{d}{dz}e^z = \frac{\partial}{i\partial y}\left[\quad\downarrow\quad\right] = -i\,e^x(-\sin y + i\cos y) = e^z \checkmark \quad (2b)$$

e^z is periodic with period 2π: $e^z = e^{z+2\pi i}$ (3)

b) trigonometric functions: These are defined in terms of the exponential function:

$$\boxed{\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \; ; \; \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)} \quad (4)$$

These are entire functions because they are expressed in terms of exponentials which are themselves entire functions. From these we have:

$$\frac{d}{dz}\sin z = \frac{1}{2i}\left(i e^{iz} + i e^{-iz}\right) = \cos z \,, \quad \text{as usual} \quad (5)$$

Other trig functions are defined as usual, but care must be taken:

$$\left.\tan z = \frac{\sin z}{\cos z}\quad\right\} \text{ analytic except when } \cos z = 0 \quad (6)$$

$\therefore$ $\tan z$ is SINGULAR at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

One can verify the C-R conditions for these functions by writing

$$\sin z \equiv u(x,y) + iv(x,y) \quad (7)$$

$$\sin z = \frac{1}{2i}\left[e^{i(x+iy)} - e^{-i(x+iy)}\right] \quad (8)$$

$$= \frac{1}{2i} e^{-y}(\cos x + i\sin x) - \frac{1}{2i} e^{y}(\cos x - i\sin x) \quad (9)$$

$$= \frac{1}{2}\left(e^{y}+e^{-y}\right)\sin x + i\frac{1}{2}\left(e^{+y}-e^{-y}\right)\cos x \quad (10)$$

$$\boxed{\therefore \sin z = \cosh y \sin x + i\sinh y \cos x = u(x,y) + i v(x,y)} \quad (11)$$

By inspection we see that the C-R conditions hold for $\forall\, x,y$

Note also that
a) $\sin iy = i\sinh y$ (12a)
b) $\overline{\sin z} = \sin \bar{z}$ (12b)
c) $\sin(z+2\pi) = \sin z$ (12c)
d) $\sin^2 z + \cos^2 z = 1$ (12d)

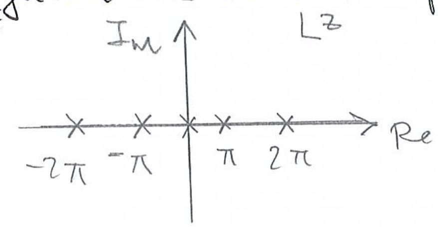
<u>Problem:</u> Find the singularities of $\csc z = \dfrac{1}{\sin z}$

<u>Solution:</u> Singularities occur when $\sin z = 0$

$$\sin z = \underbrace{\cosh y}_{\neq 0} \underbrace{\sin x}_{x = n\pi = 0, \pm\pi, \pm 2\pi, \cdots} + i\sinh y \cos x$$

for values where $\sin x = 0$ $\cos x \neq 0$ $\Rightarrow \sinh y = 0 \Rightarrow y = 0$

Hence the singularities in the complex plane are at $x = \pm n\pi,\ y = 0$:



<u>Difference from the Real case:</u> Eq. (11) $\Rightarrow |\sin z|^2 = \cancel{\cos}\,(1-\cos^2 x)(1+\sinh^2 y) + \cos^2 x \sinh^2 y$

$$= \underbrace{1-\cos^2 x}_{\sin^2 x} + \sinh^2 y - \cos^2 x \sinh^2 y + \cos^2 x \sinh^2 y$$

$$\boxed{\Rightarrow |\sin z|^2 = \sin^2 x + \sinh^2 y \Rightarrow \text{not bounded} \Big\}\ \text{general theorem!}}$$

The Complex Logarithmic Function (Intro → BRANCHES)

The multivaluedness of the angle θ begins to raise problems here.

Write $z = re^{i\theta}$ or

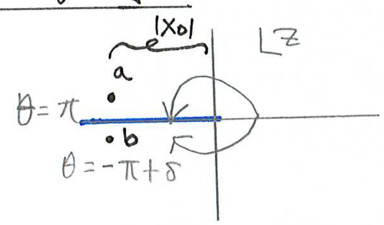
$$z = re^{i(\theta \pm 2n\pi)} \qquad n = \text{integer} \qquad (1)$$

$$\boxed{\log z = \log\left[re^{i(\theta \pm 2n\pi)}\right] \equiv \log r + i(\theta \pm 2n\pi)} \qquad (2)$$

Problem: Different values of n will lead to different numerical values for the imaginary part of $\log z$, which is therefore multivalued

Principal Value of $\log z$: take $n \equiv 0$ and $-\pi < \theta \leq \pi$

"discontinuity" of 2π across neg x-axis



← like crossing the "international date line"

↳ for the point $\underline{a}$ as shown: $\log z = \log|x_0| + i\pi$ (3)

for the point $\underline{b}$ as shown $\log z = \log|x_0| - i\pi$

However $\log z$ is defined there is a ray extending from $z=0$ to $z=\infty$ along which $\log z$ is not defined, and where it has no derivative. Elsewhere we have

$$\log z = \log r + i\theta = \underbrace{\log\sqrt{x^2+y^2}}_{u} + i\underbrace{\tan^{-1}\frac{y}{x}}_{v} \qquad (4)$$

Then $$\frac{d}{dz}\log z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{(-y)}{x^2+y^2} = \frac{x-iy}{x^2+y^2} \qquad (5)$$

$x^2+y^2 \to z\bar{z}$

$$\therefore \boxed{\frac{d}{dz}\log z = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}} \qquad (6) \qquad [\text{when } z \neq 0]$$

Side Comment

$$\log z = \underbrace{\log\sqrt{x^2+y^2}}_{u(x,y)} + i\,\underbrace{\tan^{-1}\frac{y}{x}}_{v(x,y)}$$

If you forget how to differentiate $\tan^{-1} y/x$ or $\tan^{-1}x$ or $\tan^{-1}y$ just recall that $\tan^{-1}$ is the Im part of the same analytic function of which $\log\sqrt{x^2+y^2}$ is the Re part. Since it is easy to remember how to differentiate the $\log\sqrt{\cdots}$ one can use the C-R conditions to help us remember:

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} \overset{C-R}{=} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log\sqrt{x^2+y^2} = \frac{x}{x^2+y^2} \quad \text{etc.}$$

Check on Analyticity:

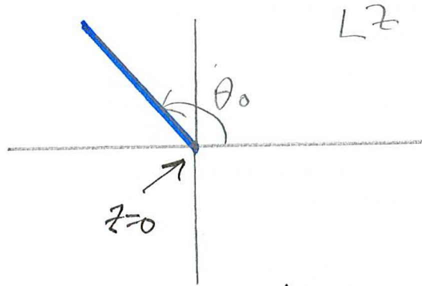
$$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2} \;;\; \frac{\partial v}{\partial y} = \frac{1}{(1+y^2/x^2)}\,\frac{1}{x} = \frac{x}{x^2+y^2} \checkmark$$

Symmetry

$$\frac{\partial v}{\partial x} = \frac{1}{(1+y^2/x^2)}\,(-y/x^2) = \frac{-y}{x^2+y^2} \;;\; -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2} \checkmark$$

BRANCHES, BRANCH CUTS & BRANCH POINTS

$\log z$ can be made <u>single-valued</u> by choosing any ray (defined by θ_0) along which we restrict θ: Pictorially



$$\boxed{\theta_0 \le \theta < \theta_0 + 2\pi} \qquad (1)$$

Each value of θ_0 defines a <u>branch</u> of $\log z$; A branch $F(z)$ of a multi-valued function $f(z)$ is any single-valued function in some domain where $F(z)$ coincides with $f(z)$. The choice $\theta_0 = -\pi$ defines the <u>principal</u> <u>branch</u>. The point $z=0$, which is common to all branches is called a <u>branch point</u>. The branch point is a <u>singular point</u> of the function $\log z$, as is every point along the ray defining the function; this ray is called a <u>branch cut</u>. At a singular point, the function is not well defined. Away from these singular points we can deal with $\log z$ as an analytic function. Thus:

$$e^{\log z} = e^{[\log r + i(\theta \pm 2n\pi)]} = \underbrace{e^{\log r}}_{r} e^{i\theta} \underbrace{e^{\pm i2n\pi}}_{1} = re^{i\theta} = z \checkmark \qquad (2)$$

Also: z^c (z, c are both complex) $\equiv [e^{\log z}]^c = e^{c\log z}$

$$\boxed{z^c \equiv e^{c\log z}} \Rightarrow (i)^i = e^{i\log i} = e^{i[\overset{0}{\log 1} + i(\overset{\pi/2}{\theta} \pm 2n\pi)]} \qquad (3)$$

$$\therefore (i)^i = e^{-\frac{\pi}{2}} e^{\mp 2n\pi} \Rightarrow \boxed{(i)^i \xrightarrow{\text{Principal Value}} e^{-\pi/2}} \qquad (4)$$

- Note that we are here expressing the function z^c in terms of e^z and $\log z$, so once we understand their analytic properties we can determine the analytic properties of other functions

- Note also that $\dfrac{d}{dz} z^c = \dfrac{d}{dz}\left[e^{c\log z}\right] = e^{c\log z} \cdot \dfrac{c}{\underset{\hookrightarrow e^{\log z}}{z}}$ (5)

$$= c\,\frac{e^{c\log z}}{e^{\log z}} = c\,e^{(c-1)\log z} = c\,z^{c-1} \checkmark \quad (6)$$

<u>Analysis of $f(z) = z^{1/2}$</u>; using $e^{c\log z} = z^c$ we have:

$$z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\ln r + i\theta)} \Rightarrow \underbrace{e^{\frac{1}{2}\ln r}}_{r^{1/2}}\, e^{i\theta/2}\, e^{i\frac{1}{2}(\pm 2n\pi)} \quad (7)$$

$$\boxed{z^{1/2} = r^{1/2}\, e^{i\theta/2}\, e^{\pm i\pi n}} \quad (8)$$

$$\boxed{\therefore\ z^{1/2} = r^{1/2}\, e^{i\theta/2}\, (-1)^n = \pm r^{1/2} e^{i\theta/2}} \quad (9)$$

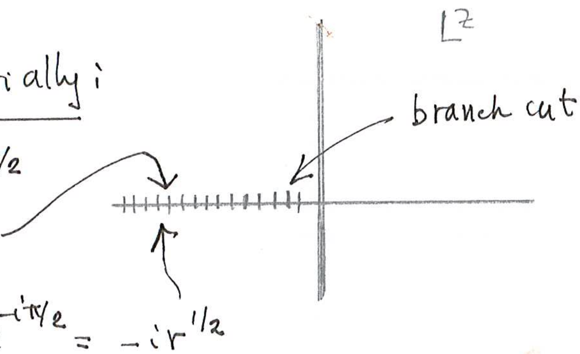
So $z^{1/2}$ is [illegible] <u>double-valued</u> just like $\sqrt{4} = \pm 2$ is double-valued. For an arbitrary z we then have 2 branches:

<u>Principal branch</u>: $z^{1/2} = r^{1/2} e^{i(\frac{\theta}{2} \pm n\pi)} \equiv f_1 \qquad -\pi < \theta \leq \pi$

<u>Other branch</u>: $z^{1/2} = r^{1/2} e^{i(\frac{\theta}{2} \pm n\pi)} \equiv f_2 \qquad -\pi \leq \theta \leq \pi$

<u>Pictorially</u>:

$z^{1/2} = r^{1/2} e^{i\pi/2} = i r^{1/2}$



$z^{1/2} = r^{1/2} e^{-i\pi/2} = -i r^{1/2}$

<u>Discontinuity</u> (in phase) $= \pi/2 - (-\pi/2) = \pi \checkmark$

INVERSE TRIGONOMETRIC FUNCTIONS:

Consider $W = \sin^{-1} z$; To study the analytic properties of $\sin^{-1} z$ we seek to express W in terms of $\log z, \ldots$ whose analytic properties we know. To do this write:

$$W = \sin^{-1} z \Rightarrow \sin W = z = \frac{e^{iW} - e^{-iW}}{2i} \Rightarrow e^{iW} - e^{-iW} = 2iz \qquad (1)$$

mult by e^{iW}

$$\therefore\ e^{2iW} - 2ize^{iW} - 1 = 0 \Rightarrow (e^{iW})^2 - 2iz(e^{iW}) - 1 = 0 \qquad (2)$$

$$\xi^2 - 2iz\xi - 1 = 0$$

$$\therefore\ \underset{\substack{\uparrow \\ e^{iW}}}{\xi} = \frac{2iz \pm (4 - 4z^2)^{1/2}}{2} = iz \pm (1 - z^2)^{1/2} \qquad (3)$$

choose + root

$$\text{Hence}\quad e^{iW} = iz \pm (1 - z^2)^{1/2} \Rightarrow \boxed{W = W(z) = -i \log\left[iz \pm (1 - z^2)^{1/2}\right]} \qquad (4)$$

$$\boxed{= \sin^{-1} z}$$

In this way we replace an "unknown" function of a simple argument with a known function of a more complicated argument. We can then use this relation to differentiate $W(z)$:

$$\frac{d}{dz} W(z) = \frac{d}{dz} \sin^{-1} z = \frac{-i}{iz \pm (1 - z^2)^{1/2}} \left\{ i \pm \tfrac{1}{2}(1 - z^2)^{-1/2}(-2z) \right\}$$

$$\text{choosing + root} \Rightarrow \{\ldots\} \Rightarrow \boxed{\frac{d}{dz} = \frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}}$$

In an analogous way one can show that

$$\boxed{\tan^{-1} z = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right)}$$

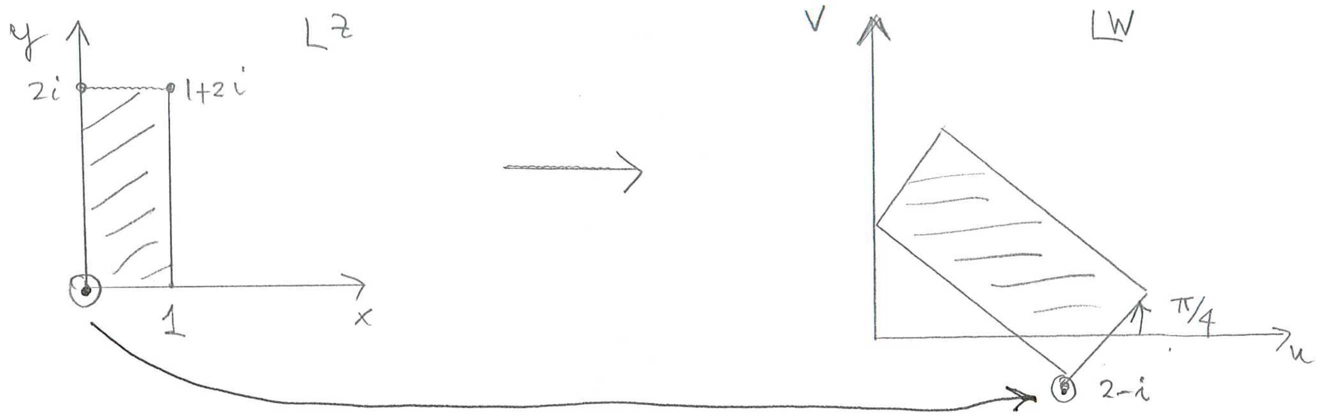
AN EXHIBITION OF PICTURES: GRAPHING (PICTURING) COMPLEX FUNCTIONS

To develop a physical picture of what $W = f(z)$ does, we can focus on a mapping of a portion of the LZ plane

a) $W = f(z) = Bz + c$ ← translates

↑ rotates & multiplies

Example: $W = u + iv = (1+i)z + (2-i)$



Note that $\arg(1+i) = \pi/4$:

↳ $re^{i\theta} \Rightarrow \theta = \arg(1+i) = \pi/4$ $\quad r = |1+i| = \sqrt{2}$

Summary: The rectangle shown in the complex z plane is first translated so that the origin → $2-i$; then it is rotated by $\pi/4$, and the lengths of the sides are multiplied by $\sqrt{2}$. All this follows immediately by writing

$$(1+i) = |1+i|\, e^{i\tan^{-1} 1} = \sqrt{2}\, e^{i\pi/4}$$

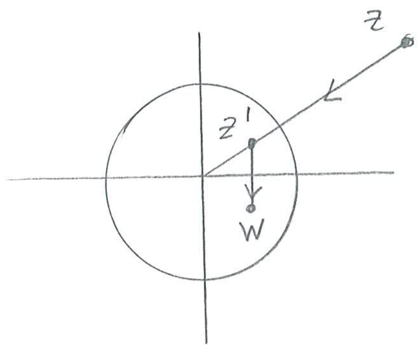
b) $w = f(z) = 1/z \equiv \rho e^{i\vartheta}$ (in w-plane) (1)

$z = re^{i\theta} \Rightarrow \frac{1}{z} = \frac{1}{r} e^{-i\theta}$ (2)

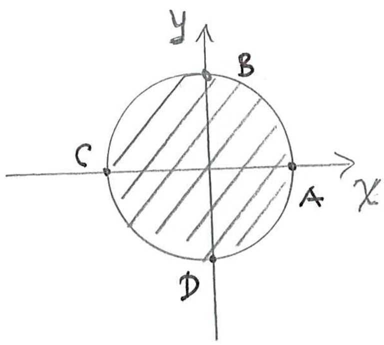
Hence (1) & (2) $\Rightarrow$ $\boxed{\rho e^{i\vartheta} = \frac{1}{r} e^{-i\theta}}$ (3)

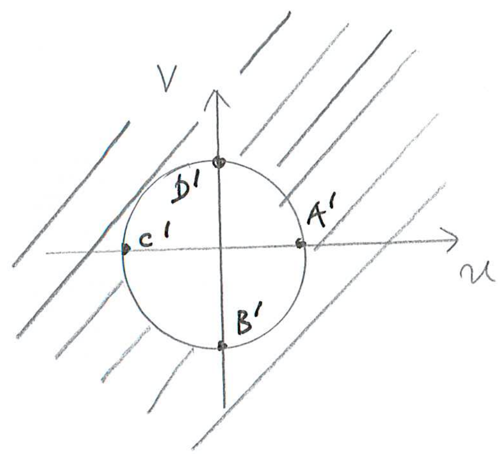
This can be pictured as the net effect of 2 successive transformations:

$$\underbrace{z' = \frac{1}{r} e^{i\theta}}_{\text{inversion with respect to unit circle}} \rightarrow w = \bar{z}' = \frac{1}{r} e^{-i\theta}$$



This maps the inside of the circle to the outside & vice versa (see below)





Note that points along the unit circle map into the unit circle except that they are inverted relative to the x-axis:

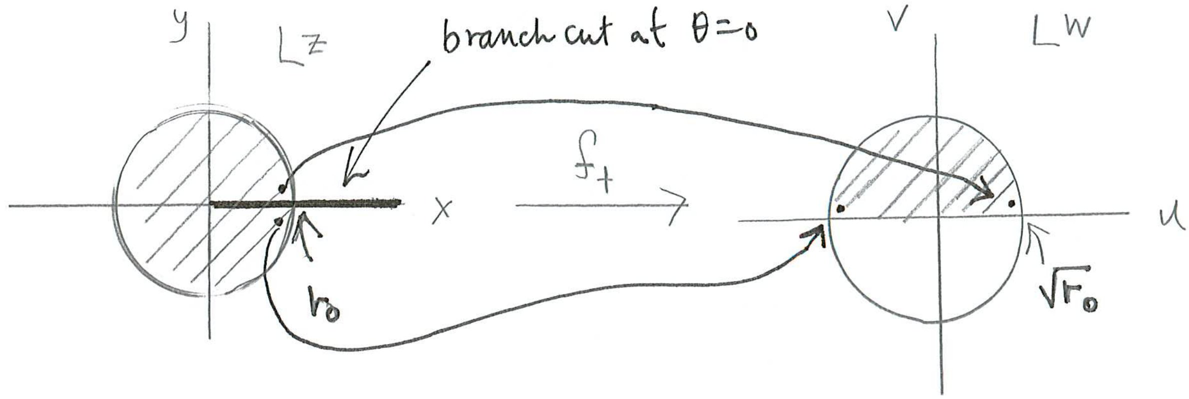
upper half plane $\rightarrow$ lower half plane.

Note also that $w = 0 \Leftrightarrow z = \infty$. This means that the transformation $w = 1/z$ is useful in studying the $z \to \infty$ limit of a complex function.

Example: $f(z) = w = \dfrac{4z^2}{(1-z)^2} \Rightarrow$ $z = \infty \Leftrightarrow w = 4$

$z = 1 \Leftrightarrow w = \infty$

c) $f(z) = w = z^{1/2}$



As noted previously, there are 2 branches $f_\pm$ to this function

$$f_\pm = \pm\sqrt{r}\, e^{i\theta/2} \qquad (0 \le \theta < 2\pi)$$

Hence a point on or just above the real axis maps to a point in a similar location as shown. However, a point just below the real axis in the LZ plane maps to a point along the negative real axis as shown. This is again a reflection of the discontinuity that arises from the presence of the branch cut.

Note that wherever the branch cut is taken to be, there will be some similar discontinuity.