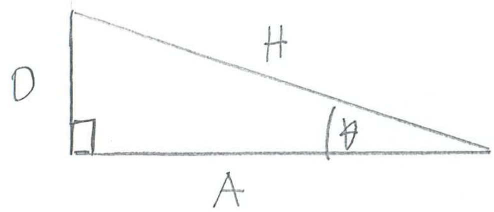
TRIG FUNCTIONS & THEIR INVERSES



Pythagoras: $A^2 + O^2 = H^2$ (1)

This can be realized by defining functions $\sin\theta$ & $\cos\theta$ so that:

$$O/H \equiv \sin\theta \Rightarrow O = H \cdot \sin\theta \,; \quad A/H = \cos\theta \Rightarrow A = H \cdot \cos\theta \qquad (2)$$

Then Pythagoras $\Rightarrow$ $H^2 \sin^2\theta + H^2\cos^2\theta = H^2 \Rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$ (3)

Having previously defined $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$ we can then define

$$\cos\theta \equiv \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \Rightarrow \cos^2\theta = \frac{1}{4}\left(\cancel{e^{2i\theta}} + \cancel{e^{-2i\theta}} + 2\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \sin^2\theta = \frac{-1}{4}\left(\cancel{e^{2i\theta}} + \cancel{e^{-2i\theta}} - 2\right)$$

add these (4)

$$(4) \Rightarrow \sin^2\theta + \cos^2\theta = 1 \; ✓$$

Note that this formula must hold for any system of units. However, when we define $\cos\theta$ & $\sin\theta$ in this way, θ is in radians. These can be checked by noting from (4) that

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \underbrace{e^{i\pi} = -1}_{\text{EULER}} = \underbrace{\cos(\pi)}_{-1} + i\,\underbrace{\sin(\pi)}_{0} \; ✓ \qquad (5)$$

$\leftarrow 180° \rightarrow$

The series expansions for $\cos\theta$ & $\sin\theta$ then follow from (4):

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} \qquad (6)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \qquad (7)$$

COMMENT: The technique of defining functions in terms of exponentials can be extended so that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ can be applied and used when

x = complex, matrix, quantum mechanical operator, ...

In the latter case, care must be taken to define what an ∞ series of matrices or operators means. More on this question later.

DERIVATIVES: Given the series expansions, we can now use those results to find the derivatives of $\sin x$ & $\cos x$.

(Henceforth the arguments of sin & cos will be assumed to be in radians, unless otherwise stated)

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\lim(\Delta x \to 0)\{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x\}}{\Delta x} \qquad (8)$$

From the previous series solutions in (6) & (7) we see that when Δx is very small $\cos(\Delta x) \simeq 1 - \frac{1}{2}(\Delta x)^2 \simeq 1$, $\sin\Delta x \simeq \Delta x$

(recall our previous result that $\frac{\sin\theta}{\theta} \xrightarrow{\theta \simeq 0} 1 \Rightarrow$

Hence (8) $\Rightarrow$

$$\therefore \frac{d}{dx}\sin x \simeq \lim_{\Delta x \to 0} \frac{\cancel{\sin x}\cdot 1 + \cos x \cdot \Delta x - \cancel{\sin x}}{\Delta x} = \cos x \qquad (9)$$

Similarly,
$$\frac{d}{dx}\cos x = \lim_{\Delta x \to 0}\left\{\frac{\cos(x+\Delta x) - \cos x}{\Delta x}\right\} = \lim_{\Delta x \to 0}\left\{\frac{\cos x \cos\Delta x - \sin x \cdot \sin\Delta x - \cos x}{\Delta x}\right\}$$

$$\therefore \frac{d}{dx}\cos x \simeq \lim_{\Delta x \to 0}\left\{\frac{\cos x \cdot (\cancel{1} - \frac{\Delta x^2}{2}) - \sin x \cdot \Delta x - \cancel{\cos x}}{\Delta x}\right\} = -\sin x \qquad (10)$$

Inverse Trig Functions:

Define $y(x) = \sin^{-1}(x)$ (1)

"y is the angle whose sine is x" $\Rightarrow \sin y = \sin(\sin^{-1}(x)) = x$

To find $\dfrac{dy}{dx} \approx \dfrac{\Delta y}{\Delta x}$ we write

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} \; ; \quad x = \sin(y) \Rightarrow \frac{dx}{dy} = \cos(y) \qquad (2)$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \qquad (3)$$

Hence: $\boxed{y(x) = \sin^{-1}(x) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} \; ;}$ (4)

Similarly: $\left.\begin{array}{l} y(x) = \cos^{-1}(x) \\ x = \cos y \end{array}\right\} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{dx/dy} = \dfrac{-1}{\sin y} = \dfrac{-1}{\sqrt{1-\cos^2 y}} = \dfrac{-1}{\sqrt{1-x^2}}$ (5)

Hence $\boxed{y(x) = \cos^{-1} x \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-x^2}}}$ (6)

Having introduced e^x and $\ln x$ we can derive a useful explicit expression for $\sin^{-1}(x)$, which holds for an arbitrary complex argument z. Start with

$$\underline{z = \sin\theta} = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \qquad < \; e^{i\theta} \equiv y \qquad (7)$$

$$e^{i\theta} \cdot 2iz = e^{2i\theta} - 1 \Rightarrow e^{2i\theta} - 2iz\, e^{i\theta} - 1 = 0 \qquad (8)$$

$$y^2 - 2izy - 1 = 0$$

$$\therefore \; y = e^{i\theta} = \frac{2iz \pm \sqrt{-4z^2+4}}{2} = iz \pm \sqrt{1-z^2} \qquad (9)$$

Choosing + [redacted] root $\Rightarrow$

Taking $\ln(\cdots)$ of both sides of (9) $\Rightarrow$

$$\ln(e^{i\theta}) = i\theta = \ln\left(iz + \sqrt{1-z^2}\right) \qquad (10)$$

But $z = \sin\theta \Rightarrow \theta = \sin^{-1}(z) \Rightarrow$ $\boxed{\sin^{-1}(z) = -i\ln\left(iz+\sqrt{1-z^2}\right)}$ (11)

Using this formula we can directly check the previous result for the derivative of $\sin^{-1}(z)$. Let $u(z) = \left(iz + \sqrt{1-z^2}\right)$

Then
$$\frac{d}{dz}\sin^{-1}(z) = -i\,\frac{d\ln(u)}{du}\cdot\frac{du}{dz} = -i\cdot\frac{1}{\left(iz+\sqrt{1-z^2}\right)}\,\frac{d(\ldots)}{dz} \qquad (12)$$

$$\frac{d(\ldots)}{dz} = i + \frac{1}{\cancel{2}}\,\frac{1}{\sqrt{1-z^2}}(-\cancel{2}z) = \frac{i\sqrt{1-z^2} - z}{\sqrt{1-z^2}} \qquad (13)$$

Combining (12) & (13) we find

$$\frac{d}{dz}\sin^{-1}(z) = \frac{-i}{\left(iz+\sqrt{1-z^2}\right)} \times \frac{\left(i\sqrt{1-z^2} - z\right)}{\sqrt{1-z^2}} = \frac{1}{\sqrt{1-z^2}} \qquad (14)$$

Which is the same result found previously.

HYPERBOLIC FUNCTIONS

$$\cosh x \equiv \frac{1}{2}(e^x + e^{-x}) \; ; \; \sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad (1)$$

$$\frac{d}{dx}\cosh x = \frac{1}{2}(e^x - e^{-x}) = \sinh x \qquad (2)$$

$$\frac{d}{dx}\sinh x = \frac{1}{2}(e^x - (-)e^{-x}) = \cosh x \qquad (3)$$

$$\cosh^2 x - \sinh^2 x = 1 \; ; \; \sinh(x+y) = \sinh x \cdot \cosh y + \cosh x \cdot \sinh y \qquad (4)$$

Inverse Hyperbolic Functions

In analogy to sin x & cos x we write

$$y = \sinh^{-1} x \iff x = \sinh y$$

Using the "inverse function" trick we have:

$$\frac{dx}{dy} = \frac{d}{dy}\sinh y = \frac{d}{dy}\frac{1}{2}(e^y - e^{-y}) = \cosh y \qquad (5)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} \xrightarrow{(4)} \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}} \qquad (6)$$

Hence $\boxed{y = \sinh^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}}$ (7)

Similarly: $y = \cosh^{-1} x \Rightarrow x = \cosh y \Rightarrow \frac{dx}{dy} = \sinh y$ (8)

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}} \qquad (9)$$

In a similar manner we can define the hyperbolic tangent function ≡ tanh x as

$$\boxed{\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}} \quad (10)$$

This function is interesting in MACHINE LEARNING because of its limits:

$$\tanh(-\infty) = \frac{\overbrace{e^{-\infty}}^{\to 0} - e^{+\infty}}{e^{-\infty} + e^{+\infty}} \cong \frac{-e^{+\infty}}{e^{+\infty}} = -1$$

$$\tanh(+\infty) = \frac{e^{\infty} - e^{-\infty}}{e^{\infty} + e^{-\infty}} \cong \frac{e^{\infty}}{e^{\infty}} = +1 \quad (11)$$

$$\tanh(0) = 0$$

Hence this function maps the whole real line (from $-\infty$ to $+\infty$) into the narrow range $[-1, 1]$, for any parameter of interest. This allows one to study different parameters that might describe the performance of a car engine (for example) on a common footing. For example, we might want to study the fuel efficiency of an engine as a function of bore, stroke, and compression ratio.

CONNECTION BETWEEN TRIG (CIRCULAR) FUNCTIONS AND HYPERBOLIC FUNCTIONS

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \Rightarrow \sin(ix) = \frac{1}{2i}\left(e^{i \cdot ix} - e^{-i(ix)}\right) \quad (1)$$

$$= \frac{1}{2i}\left(\underbrace{e^{-x} - e^{x}}_{-2\sinh x}\right) \Rightarrow \boxed{\sinh x = -i \sin(ix)} \quad (2)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \Rightarrow \boxed{\cos(ix) = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \cosh x} \quad (3)$$

$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \boxed{\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i \sinh x}{\cosh x} = i \tanh x} \quad (4)$$

We can similarly show that

$$\boxed{\begin{aligned} \coth x &= i \cot(ix) \\ \operatorname{sech} x &= \sec(ix) \\ \operatorname{csch} x &= i \csc(ix) \end{aligned}}$$

- 25.2

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cosh x\right) = \sinh x$$

$$\frac{d}{dx}\left(\sinh x\right) = \cosh x$$

$$\frac{d}{dx}\left(\tanh x\right) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\left(\coth x\right) = -\csc h^2 x$$

$$\frac{d}{dx}\left(\operatorname{sech} x\right) = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx}\left(\operatorname{csch} x\right) = -\operatorname{csch} x \cdot \coth x$$

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

GRAPHING FUNCTIONS

To have a physical feeling for a mathematical function it is optimally helpful to graph it. Although this can always be done numerically, insight can usually be obtained by considering various limiting cases of some variable ($x=0, x\to\pm\infty \ldots$) and also by examining its derivatives.

The previously discussed function $\underbrace{\tanh(x)}_{\equiv y(x)}$ is an example. We have already shown that $\tanh(0)=0$ and $\tanh(\pm\infty)=\pm 1 \equiv y(\pm\infty)$

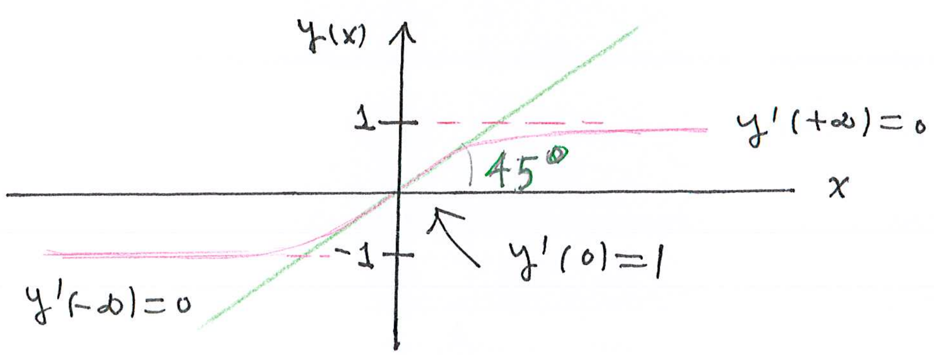
Further insight can be had by looking at its derivative:

$$\frac{d}{dx}\underbrace{\tanh(x)}_{y(x)} = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{(\cosh x)^2}$$

$$= 1/(\cosh x)^2 = \frac{1}{\left[\frac{1}{2}(e^{x}+e^{-x})\right]^2} \qquad (1)$$

$$\Rightarrow \frac{d}{dx}\underbrace{\tanh(x)}_{y(x)} = \frac{4}{(e^{x}+e^{-x})^2} \equiv y'(x) \Rightarrow \begin{array}{l} y'(+\infty)=y'(-\infty)=0 \\ y'(0)=1 \end{array} \qquad (2)$$

Combining these results with the previous results, $y(\pm\infty)=\pm 1$ we can sketch $y(x)=\tanh(x)$ as follows



[EXAMPLE 2] $y = f(x) = x^n e^{-x}$ $\quad (x \geq 0,\ n \geq 1)$

We see immediately that: $y(0) = 0$; $y(\infty) = 0$

$$y'(x) = x^n \Big\{ \underbrace{\frac{d}{dx} e^{-x}}_{-e^{-x}} \Big\} + e^{-x} \Big\{ \underbrace{\frac{d}{dx} x^n}_{nx^{n-1}} \Big\} = e^{-x} \{ -x^n + nx^{n-1} \} \qquad (1)$$

$$\Rightarrow y'(x) = e^{-x} \cdot x^{n-1} \{ -x + n \} \Rightarrow \underline{y'(x) = 0 \text{ when } x = n} \qquad (2)$$

Recall that $y'(x) = 0$ can signify either a maximum or a minimum. To determine which it is we evaluate the 2ND derivative $y''(x)$ at $x = n$: From (1)

$$y''(x) = -e^{-x} \{ -x^n + nx^{n-1} \} + e^{-x} \{ -nx^{n-1} + n(n-1)x^{n-2} \}_{x=n}$$

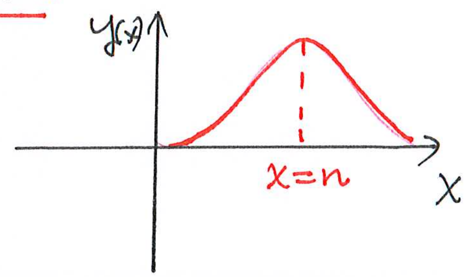
$$= e^{-x} \{ x^n - nx^{n-1} - nx^{n-1} + n(n-1)x^{n-2} \}_{x=n}$$

$$= \underbrace{e^{-x} (x^{n-2})}_{\text{positive}} \{ x^2 - 2nx + n(n-1) \}$$

when $x = n \Rightarrow \{\ldots\} = n^2 - 2n^2 + n(n-1)$
$= -n < 0$

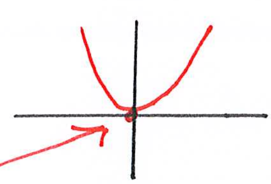
Since $y''(x = n) < 0 \Rightarrow$ <u>maximum.</u>*

Hence $y = x^n e^{-x}$ looks like



* USEFUL MNEMONIC:

Consider $y = x^2$



$y'(x) = 2x = 0$
$\Rightarrow x = 0$

$y'' = 2 \Rightarrow y''(0) = 2 = \text{positive} \Rightarrow$ $\quad y'' = \text{positive} \Rightarrow \text{minimum}$
$\quad y'' = \text{negative} \Rightarrow \text{maximum}$

[Example 3] From the text p. 23

$$y = f(x) = \frac{(x^2 - 5x + 6)}{x - 1} \cdot e^{-x/5} \qquad (1)$$

Step [1]: Examine $x \to +\infty$; We immediately note that $e^{-\infty/5} \to 0$.
Since $e^{-x/5}$ contains all powers of x (recall $e^y \equiv \sum_{n=0}^{\infty} y^n/n!$)
the behavior of $f(x)$ as $x \to \infty$ is dominated by $e^{-x/5}$

$\Rightarrow f(x) \to 0$ as $x \to \infty$

Step [2]: Examine the behavior of $f(x)$ as $x \to -\infty$. In this case

$$e^{-x/5} \longrightarrow e^{|x|/5} \to \infty \qquad (2)$$

At the same time the rational function multiplying $e^{-x/5}$ is dominated by x^2 in the numerator, and by x in the denominator

Hence $$f(x) \xrightarrow{x \to -\infty} \frac{x^2}{x} e^{|x|/5} = x e^{|x|/5} \qquad (3)$$

Step [3]: Examine $f(x)$ for other finite values of x.

Here we note that (by inspection) $f(x)$ can be rewritten in the form

$$f(x) = \frac{(x-2)(x-3)}{x-1} e^{-x/5}$$

Hence $f(x=2) = 0$ and $f(x=3) = 0$, but $f(x=1) \longrightarrow \infty$

For homework you will be asked to compute $f'(x)$, and then use that result to reproduce Figure 1.6 of the text.

[EXAMPLE 4] Study the function $\boxed{y = f(x) = x^n \ln x}$

$n \geq 1$ is an integer

Here the behavior as $x \to +\infty$ is obvious: $f(x) \to +\infty$

The interesting question is what happens as $x \to 0$. Bear in mind that $\ln(0) \to -\infty$.

A convenient way to analyze y is to substitute $z = 1/x$ so that $x \to 0$ becomes $z \to \infty$. Then

$$f \to \frac{1}{z^n} \ln\left(\frac{1}{z}\right) = \frac{1}{z^n}(\ln 1 - \ln z) = -\frac{\ln z}{z^n} \xrightarrow{z \to \infty} \frac{\infty}{\infty} \quad (1)$$

Applying l'Hopital's Rule: $f \to \dfrac{\frac{d}{dz}(-\ln z)}{\frac{d}{dz} z^n} = \dfrac{(-1/z)}{n z^{n-1}} = -\dfrac{1}{n}\dfrac{1}{z^n} \quad (2)$

Hence as $x \to 0 \leftrightarrow z \to \infty$ $f \to 0$.

$$\boxed{\text{So } y = f(x) = x^n \ln x \xrightarrow{x \to 0} 0} \quad (3)$$

Hence the growth of $\ln x$ as $x \to \infty$ is weaker (slower) than the growth of any polynomial x^n.

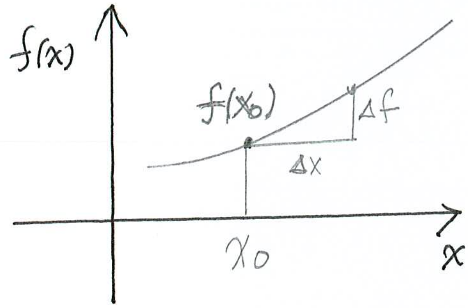
Note that in this case we could have obtained the same result by applying l'Hopital's Rule directly in the form

$$y = f(x) = x^n \ln x \xrightarrow{x \to 0} \left(\frac{d}{dx} x^n\right)\left(\frac{d}{dx} \ln x\right) = (n x^{n-1}) \frac{1}{x} = n x^{n-2} \xrightarrow{x \to 0} 0$$

But you should be careful doing this elsewhere!!

-30

DIFFERENTIALS



SEE Fig. 1.8 of text

From the figure, the change Δf in $f(x)$ from x_0 to $x_0+\Delta x$ is given by

$$\Delta f = \left.\frac{df}{dx}\right|_{x_0} (\Delta x) + \text{terms of order } (\Delta x)^2 \text{ and higher denoted by } \ldots \quad (1)$$

In the limit as $\Delta x \longrightarrow dx$, $\Delta f \to df$ and we can write

$$\boxed{df(x_0) = \left.\frac{df(x)}{dx}\right|_{x_0} \cdot dx} \quad (2)$$

As a practical matter we often use the approximation in (1) to estimate the small change Δf that will result from changing $x_0 \to x_0 + \Delta x$.